Tube Categories and 3d TQFTs

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Mathematics and Applications

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1 Introduction

This thesis is, first and foremost, about 3-dimensional oriented TQFTs. These have an already long history within the field of low-dimensional topology, and recent applications to condensed matter physics have renewed the demand for them to be carefully examined. The general ethos, when it comes to 3d-TQFTs is given by:

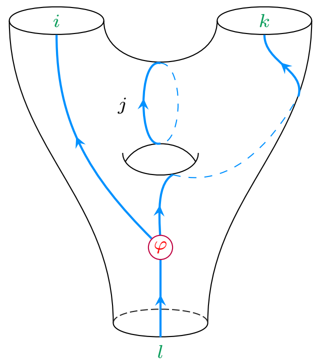
$$\text{3-dimensional TQFTs} \longleftrightarrow \text{Modular Tensor Categories} \tag{1.1}$$

A lot of the seminal work done by Kirillov Jr. and Bakalov [2] on this subject, revolves around trying to establish 1.1. Although this is a nice guiding principle, and a fruitful research slogan, it underplays a rich story. For one, we care about TQFTs that extend beyond Atiyah's program. In particular, we are interested in developing theories that have greater *locality*. By locality we mean the notion, inspired by physicists rejection of any spooky actions at a distance, that one should be able to tell what TQFT one has just by understanding its behaviour on the neighborhood of a point in spacetime, or, ideally, to uderstanding its behaviour at every point. In other words, we are interested in a class of TQFTs called *fully extended*, whose classification was proposed by Baez and Dolan under the framework of the *Cobordism Hypothesis* [1], and for which a proof has been sketeched by Lurie and Hopkins [18]. In the case of oriented TQFTs, the cobordism hypothesis establishes a 1-1 mapping between fully extended 3-dimensional TQFTs and $\mathrm{SO}(3)$-homotopy fixed-points in the 3-category of $\mathbb{C}$-linear monoidal categories — which are conjectured [6] to be *Spherical Fusion Categories* [4].

Complementing this result, in a recent paper by Bartlett-Douglas-Schommer–Pries-Vicary [5], it is proved that oriented 3d-TQFTs that are once-extended i.e. extended down to the circle, are in 1-1 correspondance with anomaly-free modular tensor categories (these are also called theories with trivial central-charge in Conformal Field Theory), examples of which are given by taking the center $\mathcal{Z}(\mathcal{C})$ of a spherical tensor category $\mathcal{C}$ (see Remark 5.19 in [21]).

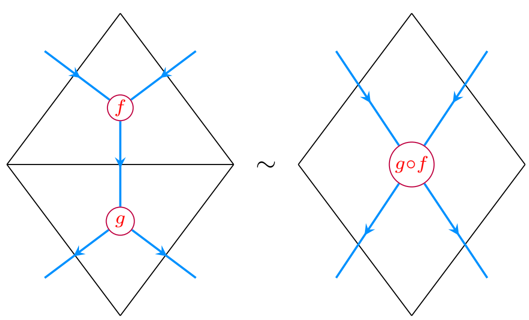
In this thesis we consider how one can take the hint from the cobordism hypthesis, by focusing on building oriented TQFTs from a spherical fusion category $\mathcal{C}$, which we will call String-Net TQFT, and showing that they correspond, when interpreted as only once-extended, to the MTCs given by taking the Drinfel'd center of $\mathcal{C}$. In the process we show that, as the cobordism hypothesis suggests, the String-Net construction is equivalent to the other famous TQFT defined using a spherical fusion category, the Turaev-Viro-Barret-Westbury state-sum [23][3], TVBW for short. The advantage here is that String-Nets are far simpler to build and interpret than the state-sum. We also try and shed light on the conjecture that fully extended TQTFs are in 1-1 correspondence with Topological Phases of Matter by proving that the Levin-Wen model, believed to capture the universal properties for these phases, is a Hamiltonian realization of the String-Net Fiel Theory.

The String-Net construction, as a TQFT, first appeared in Walker's work [24] as an example of a broader construction called TQFTs via *topological fields and local relations*. Put bluntly, as a 3-dimensional TFQT (let us call it $\mathcal{Z}$, for now), it assigns to every oriented surface Σ (without corners), a Hilbert space constructed out of all $\mathbb{C}$-linear combinations of graphs properly embedded in Σ, decorated with the data of some spherical fusion category $\mathcal{C}$. These decorated, or labelled graphs, are what Walker calls topological fields, an example of what we mean by a decorated graph is given in the picture below.



(1.2)

As we will see, the fact that we are making use of a spherical category, in particular a pivotal category, allows us to interpret the pictures inside any embedded disk as a string-diagram in $\mathcal{C}$. By converting an embedded disk, call it $D^2 \subset \Sigma$, to an equivalent string-diagram i.e. one corresponding to the same morphism in $\mathcal{C}$, we construct what are called *local relations* and we dub this operation *evaluation* in D^2. Local relations correspond to the set of equations between any pair of decorated graphs that differ only by a finite sequence of applications of evaluation map. The Hilbert space associated to Σ, is the space of all $\mathbb{C}$-linear decorated graphs in Σ, modded out by the local relations prescribed above. The fact that this vector space is a topological invariant of the surface (in the case the surface is closed) has to do with how the local relations provided by a spherical fusion category mimic the typical subdivision and mutation moves that are associated to cellular-decompositions of surfaces. A stand-out simple example to get to grips with this way of relating labelled graphs is the relation between removing an edge of the cellular-decomposition and the composition of 1-morphisms in $\mathcal{C}$:

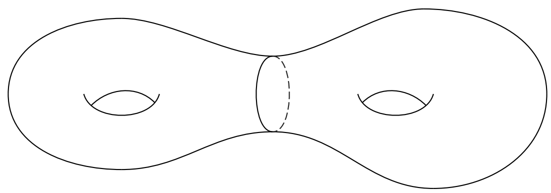


We would say that the left string-diagram is evaluated to the same string-diagram on the right. We sum up the construction for closed manifolds as follows:

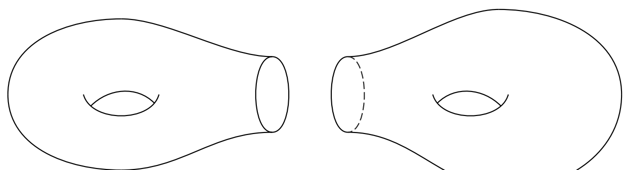
$$\mathcal{Z}(\Sigma) \; := \; \mathbb{C}\{\text{decorated graphs in } \Sigma\}/\{\text{local relations}\} \tag{1.3}$$

This naturally extends to manifolds without corners, as is suggested by the picture 2, so long as we refrain from applying any local relation carried out by an evaluation intersecting the boundary circles of the surface. This suggests that, for any given oriented surface with non-empty boundary and with an embedded labelled graph Γ there is a boundary condition that records how the labelled graph intersects the boundary. In other words, boundary conditions amount to oriented circles $B_1, \ldots, B_k$ with marked points labelled by objects of $\mathcal{C}$, which we denote as $\{V_{b_1}\}_{b_1 \in B_1}, \ldots, \{V_{b_k}\}_{b_k \in B_k}$ — we will often use short-handed versions of this notation of boundary conditions.

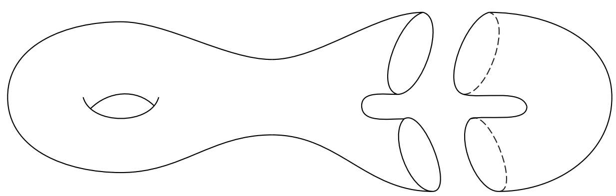
The next step is to use the fact that, according to the cobordism hypothesis, the string-net TQFT is fully-extended and in particular once-extended. This means that the vector space $\mathcal{Z}(\Sigma)$ has to be able to be computed by chopping Σ as the gluing of simpler surfaces (most likely with boundary) and then apply some gluing rules using the vector space associated to each of the simpler pieces. If Σ is the 2-holed torus



then we can write Σ as the gluing of the following two once-punctured toruses



or, alternatively, we can write it as the gluing of a cylinder with a twice-punctured torus



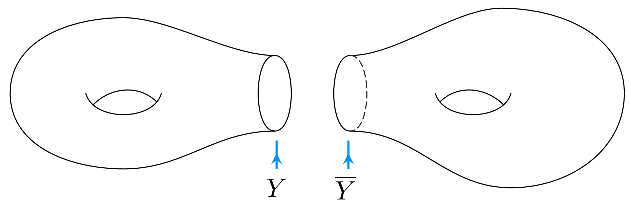
the combinations for additional splittings and regluings are quite literally endless. We have mentioned gluing rules — when our objects of study are the standard non-extended parts of a TQFT, the gluing of two bordisms is algebraically carried out by composing linear maps — and understanding them for higher-codimensional splittings involve understanding the 1-categorical data that is encoded in the boundary circles of the oriented surfaces we have been examining. That 1-categorical data is what we will call the annularization of $\mathcal{C}$ or the tube category associated to $\mathcal{C}$ — which we will call $\mathrm{Tube}_{\mathcal{C}}(S^1)$. Its objects are given as boundary conditions $\{V_b\}_{b \in B}$ with the morphisms given by

$$\mathrm{Hom}(\{V_b\}, \{W_c\}) \ := \ \mathcal{Z}\Big(\ , \{V_b\}^\vee, \{W_c\} \Big)$$

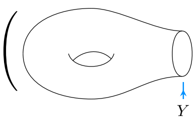
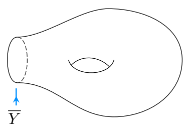
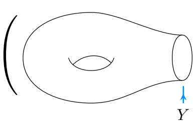
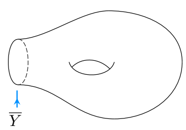
where the $\vee$ will be explained later — it is related to the fact the we consider with an incoming and an outgoing circle and composition is given by gluing (stacking tubes). And we define

$$\mathcal{Z}(S^1) := \mathrm{Tube}_{\mathcal{C}}(S^1) \tag{1.4}$$

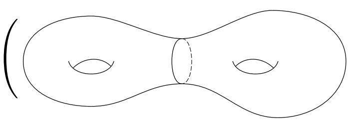
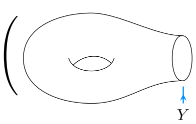
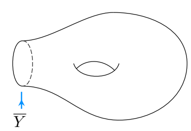
As we will explore in greater detail, by fixing a set of boundary conditions, the vector spaces associated to these surfaces afford modules for the tube category, where the action is defined by gluing string-net-colored collars to the boundary circles. Picking up the example of the two-holed torus again



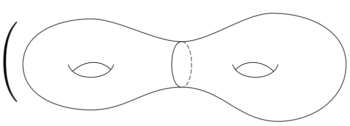
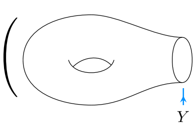
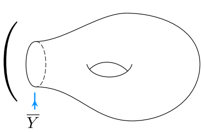
then we have that:

- $\mathcal{Z}\Big($  $\Big)$ is a $\mathrm{Tube}_{\mathcal{C}}(Y)$-module;
- $\mathcal{Z}\Big($  $,\Big)$ is a $\mathrm{Tube}_{\mathcal{C}}(-Y) = \mathrm{Tube}_{\mathcal{C}}(Y)^{\mathrm{op}}$-module;
- $\mathcal{Z}\Big($  $\bigsqcup$  $\Big)$ a $\mathrm{Tube}_{\mathcal{C}}(Y) \times \mathrm{Tube}_{\mathcal{C}}(Y)^{\mathrm{op}}$-module.

what we manage then to prove is that:

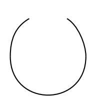
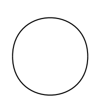
$\mathcal{Z}\Big($  $\Big) = \mathrm{Coend}\Big[\mathcal{Z}\Big($  $\bigsqcup$  $\Big)\Big]$

or as the coend is more often referred to, when applied to the language of representations:

$\mathcal{Z}\Big($  $\Big) = \mathcal{Z}\Big($  $\Big) \underset{\mathrm{Tube}_{\mathcal{C}}(Y)}{\otimes} \mathcal{Z}\Big($  $\Big)$

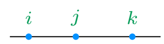
it corresponds to the tensor product of a left and right module of $\mathrm{Tube}_{\mathcal{C}}(Y)$. This provides the gluing rule (or schema) for gluing along codimension-1 manifolds without corners. This is an extremely useful result as we will see, because it introduces a formula that we can leverage to understand how data, encoded in the vector space of a surface, might be localized to a particular region. Moreover, it introduces a greater algebraic understanding to the cut and paste nature of string-diagrams and is backed by a vast literature on (co)ends.

This construction can be generalized in different ways. The most extensive, and laborious one to achieve, is to keep going down the categorical loop-hole and to attribute a 2-category to the point. In our construction this 2-category is the bicategory $\mathcal{C}$ with a unique object, meaning that the point is always attributed the same algebraic structure i.e. $\mathcal{C}$. This is not the laborious aspect of the construction, that would be understanding how to glue 1-manifolds along their corners. For example:

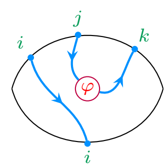
$Y =$  $\xrightarrow{\text{glue}}$ $Y_{\text{glued}} =$ 

For the example above, any gluing formula would involve a general understanding of birepresentation theory. This is a consequence of the fact that, as opposed to what happens with surfaces, where the algebraic objects are vector spaces here the glued manifold is really a 1-manifold, implying that our categorified gluing needs to recapture the 1-category associated to the circle as a sort of categorified coend, 2-coend or 2-tensor product over the bicategory associated to the point. The other generalization route is to consider surfaces with corners — the reason why this pursuit ends up being not as complicated as the one mentioned above, is twofold: first, we are still just gluing surfaces along 1-manifolds, and hence we dispense the need for higher categorical tools, moreover since our 2-category is really just the delooping of a 1-category, our corners are handled rather trivially.

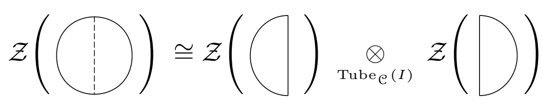
We defined the tube category as the category whose objects are boundary conditions to string-nets of surfaces with boundary i.e. circles with a decorated 1-graph. To comprehend surfaces with corners, we must extend the tube category to all 1-manifolds, therefore, for open intervals we consider objects to be decorated 1-graphs without vertices at the endings:



where the morphisms are given by string-nets on the bigon (also called pinched interval cobordism of the intreval) whose boundary is split by the source and target objects, respectively. An example of a morphism in this generalized tube category is given by the following diagram:



Just like the string-net vector spaces, for manifolds without corners, provide a module for the S^1 tube category, the string-net vector spaces for surfaces with corners provide modules for the interval tube category. And through the application of an argument *mutatis mutandis* to the cornerless case, we get that:

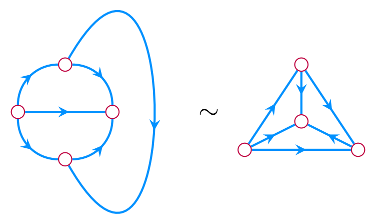


(1.5)

which offers an extension of our gluing/cut-and-paste toolkit.

Before trying to relate the string-net construction to anything else, it is useful to add a remark on the nature of spherical fusion categories. In other words, provide reasoning on why they seem to be linked to topological quantum field theories in the first place. Ever since the beginning of quantum topology, between the mid-1980's and early 1990's, particular attention was cast on the role of *Quantum Groups* as a useful algebraic gadget to construct invariants of links — now aptly named quantum invariants, whose distinguishing power extended beyond the classical tools of *knot theory* i.e. homotopy groups of the knot's complement — and as a consequence produce 3-manifold invariants subject to being constructed via a 3d-TQFT. Quantum groups, opposite to what their name might suggest, are really algebras, and in particular Hopf algebras with some additional structure and whose representation categories have the structure of spherical tensor categories.

With this historical background past us, it is sensible to state, in rough terms, why spherical fusion categories, as an intrinsically algebraic object, can be manipulated as pictures or string-diagrams. We begin by recalling that for tensor categories there is an extremely useful diagrammatic language, defined by representing objects as 1-dimensional strands and morphisms as boxes, where composition is given by contracting strands and tensoring by horizontal juxtposing whatever is being tensored. This allows one to evaluate complicated algebraic equations by manipulating these strands in topologically "sensible" ways. Spherical categories gather all the necessary algebraic data to ensure full isotopy-invariance of the aforementioned diagrammatic calculus and guarantees that any closed graph can be interpreted as an endomorphism of the monidal unit $\mathrm{End}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$ (which is an axiom, not a theorem). As such, and as an example, the following pair of graphs (we omit the labelling)



are evaluated to same morphism in $\mathrm{End}(\mathbb{1}, \mathbb{1}) \cong \mathbb{C}$ i.e. the same scalar. Hopefully, this provides a clearer picture of why these algebraic constructions are useful when probing topological properties. One of the most important properties of these categories is that any object can be written as a direct sum of what are called simple objects $X_i \in \mathrm{Irr}(\mathcal{C})$. Moreover, the list of simple objects is assumed to be finite and the morphism spaces between them must satisfy:

$$\mathrm{Hom}(X_i, X_j) \cong \delta_{i,j}\,\mathbb{C} \tag{1.6}$$

We call these semisimple categories and often refer to them as fusion categories because of the way they have been used by physicists to study fusions of particles, defects and domain walls in certain QFTs, giving rise to the name fusion algebra for the set of coefficients N_{ij}^k appearing in the decomposition into simple objects:

$$X_i \otimes X_j \cong \bigoplus_{X_k \in \mathrm{Irr}(\mathcal{C})} N_{ij}^k \cdot X_k \tag{1.7}$$

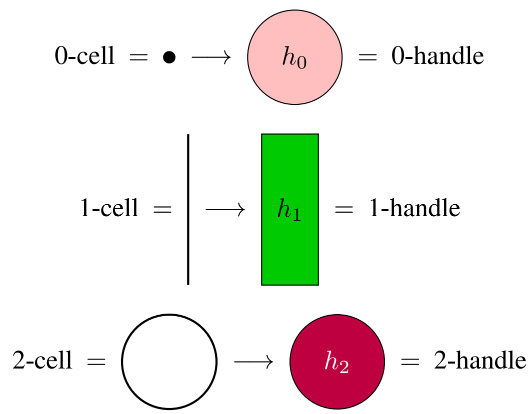
where $N_{ij}^k \cdot X_k := X_k \oplus \cdots \oplus X_k$ has N_{ij}^k components. The interplay between these fusion rules and the rest of the categorical data determine much of what we had previously called local relations. They are equations, typically written in the pictorial language we have been describing, and encapsulate structural data of the category. Examples of these categories come in varied flavours:

- Vect_k — the category of finite-dimensional vector spaces of an algebraically closed field k;
- $\mathrm{Vect}_{\mathbb{C}}[G]$ — the category of G-graded vector spaces over the complex numbers. where G is some finite group;
- $\mathrm{Rep}(G)$ — the category of finite-dimensional representations of G over the complex numbers, for some finite group G;
- $\mathrm{Rep}(U_q(\mathfrak{g}))$ — the (sub)category of (appropriately reduced to quotients by tilting modules) of representations of the universal enveloping algebra of a semisimple Lie algebra $\mathfrak{g}$, and such that q is a root of unity.

The list goes on and has many exotic examples which are still poorly understood, especially for categories with numbers of simple objects above 5, one of the great challenges being the absence of a counterpart to simple groups in finite group classification. The work on subfactors here has proved extremely useful.

Following this prelude to the first chapter on spherical fusion categories, we skip to a soft, albeit general discussion, of how to take the string-net construction to setup a sum of locally commuting projectors (a Hamiltonian), for any given oriented closed surface Σ, and for which the ground-state is canonically isomorphic to $\mathcal{Z}(\Sigma)$. This should serve as a stepping stone to the discussion of the Levin-Wen model later in the thesis, but it is meant, first and foremost, to bridge the gap between local or fully extended TQFTs and the study of physical models with emergent "topological" properties, also called topological phases of matter.

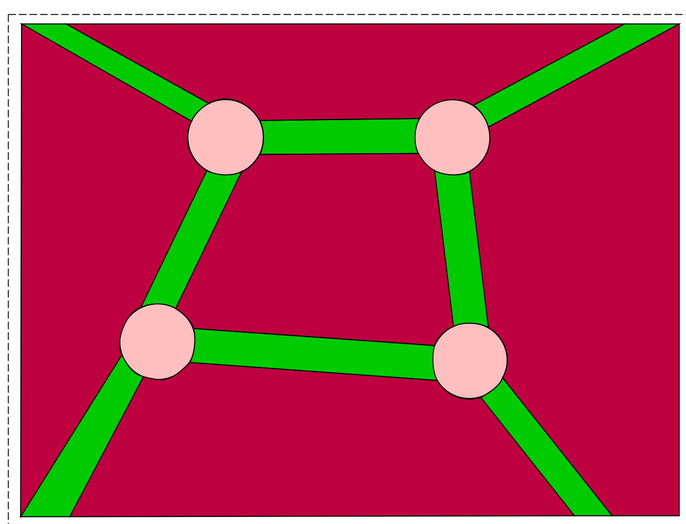
Let Σ be an oriented closed surface and let H_Σ be a handle decomposition of Σ. For the unfamiliar with the last sentence, consider instead a generic cellular decomposition of Σ and "fatten" every k-skeleton in this decomposition. For a 2-dimensional handle decomposition we are restricted to 0-cells, 1-cells and 2-cells or 0-handles, 1-handles and 2-handles. The correspondence between the two is given as follows:



More formally, a 2-dimensional k-handle is identified with a manifold homeomorphic to $h_k = D^k \times D^{2-k}$. We split its boundary

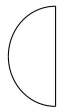
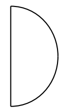
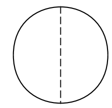
$$\partial(h_k) = \partial(D^k \times D^{2-k}) = (\partial(D^k) \times D^{2-k}) \cup (D^k \times \partial(D^{2-k})) = (S^{k-1} \times D^{2-k}) \cup (D^k \times S^{2-k-1}) \tag{1.8}$$

into what we call an attaching boundary (the set on the left-hand side of the boundary decomposition) and a non-attaching boundary (the set on the right-hand side of the boundary decomposition. The following picture is chosen to exemplify a local patch of a generic 2-dimensional handle decomposition.



Taking this handle decomposition H_Σ how can we construct the commuting projectors we are looking for? Our aim is to build a set of projectors whose images, when intersected, give us a space isomorphic to the space of string-nets in

Σ. One way to do this is achieved by taking a closer look at the gluing formulas we presented in the beginning of the introduction. Let us rewrite equation 1.5, but first establish some names:

$$\Sigma_1 := $$  $$\bigsqcup \quad \Sigma_2 :=$$  $$\xrightarrow{\text{glue}} \Sigma_1 \cup_I \Sigma_2 =$$  (1.9)

and we define I to be the interval component of the boundary along which we will glue both disks. Then equation 1.5 can be rewritten as

$$\mathcal{Z}(\Sigma_1 \cup_I \Sigma_2) \cong \left[\bigoplus_{\{V_b\} \in \mathrm{Tube}_{\mathcal{C}}(I)} \mathcal{Z}(\Sigma_1, \{V_b\}) \otimes \mathcal{Z}(\Sigma_2, \{V_b\}) \right] \Big/ \langle (\Gamma_1 \cdot \alpha) \otimes \Gamma_2 \sim \Gamma_1 \otimes (\alpha \cdot \Gamma_2) \rangle \tag{1.10}$$

where Γ_1 and Γ_2 are string-net configurations in both Σ_1 and Σ_2, respectively, and α is a string-net configuration of the bigon i.e. a morphism of $\mathrm{Tube}_{\mathcal{C}}(I)$ and the $\cdot$ is notation for the module action. Since both $\mathrm{Tube}_{\mathcal{C}}(S^1)$ and $\mathrm{Tube}_{\mathcal{C}}(I)$ are semisimple categories (see [21]), there must exist a set of finite idempotents, let us call them

$$\alpha_i \colon \{V_{b_i}\}_{b_i \in B_i} \longrightarrow \{V_{b_i}\}_{b_i \in B_i},$$

which one can use to decompose any module M into simple submodules. In other words, given the action $\{\pi_\alpha \colon M \longrightarrow M\}_{\alpha \in \mathrm{Tube}_{\mathcal{C}}(I)}$, we can decompose any module M as

$$M \cong \bigoplus_{\alpha_i} \mathrm{im}(\pi_{\alpha_i}) \quad \text{also written as} \quad M \cong \bigoplus_{\alpha_i} \pi_{\alpha_i}(M) \tag{1.11}$$

Assuming then, that we have a set of minimal idempotents for both the tube category of the circle and the interval, we can decompose 1.10 in the following manner

$$\mathcal{Z}(\Sigma_1 \cup_I \Sigma_2) \cong \bigoplus_{\alpha_i} \pi_{\alpha_i}\big(\mathcal{Z}(\Sigma_1, \{V_{b_i}\}_{b_i \in B_i})\big) \otimes \pi_{\alpha_i}\big(\mathcal{Z}(\Sigma_2, \{V_{b_i}\}_{b_i \in B_i})\big) \tag{1.12}$$

Looking back at the handle decomposition H_Σ and following the line of reasoning that lead us to formula 1.12, we can construct the vector space $\mathcal{Z}(\Sigma)$ by understanding well the Hilbert spaces $\mathcal{Z}(h_k)_{k=0,1,2}$ of each k-handle in the decomposition and the projectors associated to their corresponding attaching boundaries. In particular, for each k-handle we have a Hilbert space

$$\mathcal{H}(h_k) \cong \bigoplus_{\alpha_i \colon \{V_{b_i}\}_{b_i \in B_i} \circlearrowleft} \mathcal{Z}(h_k, \{V_{b_i}\}_{b_i \in B_i}).$$

The degrees of freedom for this space might be split in two: the ones counted by the dimension of $\mathcal{Z}(h_k, \{V_{b_i}\}_{b_i \in B_i})$ which we will call (following Walker's suggestion) "vertex" degrees of freedom and the ones counted by the number of idempotents, α_i, for the tube category in question, and we will call these (also under Walker's suggestion) "idempotent" degrees of freedom.

We can use these local Hilbert spaces, associated to every k-handle, to construct a bigger Hilbert space for the whole surface by tensoring them over all k-handles in the decomposition, for all $k = 0, 1, 2$. To construct a Hamiltoninan we still need a set of commuting projectors. These are provided by the finite set of idempotent projectors π_{α_i}, associated to the tube categories of the attaching boundary of the handles, and bookkeeping projectors that ensure that the idempotents on on both sides of the gluing region actually match. A Hamiltonian can be constructed by summing over all these different types of projectors.

For the reader more familiar with these models this might seem like an awkward method of constructing a Hamiltonian model let alone the Levin-Wen model. In the thesis, we construct the Hamiltonian from a more conventional starting point, a lattice-like structure embedded in a closed surface with "spins" or small Hilbert spaces (relative to the macroscopic Hilbert space of the system) whose Hamiltonian, composed from a finite set of commuting projectors indexed by the plaquettes and vertices of the lattice, has a ground-state naturally isomorphic to the space of string-nets. This mimics the original approach of Levin and Wen [16], which was itself inspired by the seminal work of Kitaev [14] on the Toric Code and is still the most familiar way of describing, working and constructing Hamiltonians for topological phases. Still, we think this approach provides much needed clarity to the conjecture that all fully extended TQFTs can be realized as commuting projector Hamiltonians since it sheds light in the relevancy of finding idempotents in a category and constructing more computable gluing formulas.

1.1 Outline of the Thesis

This thesis runs as follows. In Section 2 we describe both the algebraic and categorical properties underpinning the notion of a spherical tensor category. We construct them starting from the definition of -Barret-Westburymonoidal category and we provide, in parallel, a complete survey for their diagrammatic calculus in the form of string-diagrams. Moreover, we introduce, the less commonly used, notation of circular coupons for morphisms, a practice which lays as the bedrock of the rest for the material in the thesis.

In Section 3 we introduce some of the main subject of the thesis, starting with the vector space of String-Nets for an oriented surface (possibly with boundary). We introduce the notion of labeled graphs on a surface and, the all-important, idea of a local relation as the kernel of an evaluation i.e. a well-defined linear map from the big space of labeled graphs to a space of morphisms in $\mathcal{C}$. Additionally, we introduce the F-symbols of a fusion category and an important subset of local relations constructed using the knowledge of the F-symbols: the *F-move*, the *bubble removal* and the *Sputnik Killer*. We construct the tube category, as a linear 1-category associated to every oriented 1-manifold using String-Nets and we highlight its importance in further extending the TQFT. In particular, we prove that the vector spaces mentioned above serve as (left & right) modules for the tube category and we leverage that result to prove that gluing surfaces along codimension-1 manifolds is akin to taking the tensor product of modules in the appropriate category (We call this a coend. It is a categorification of many things, including the tensor product of modules over a ring). We also mention how to provide a slight generalization of the tube category associated to the circle into a tube category associated to the interval. The last subsection is focused on describing the space of String-Nets as a quotient of the vector space of String-Nets for surfaces with punctures. A characterization needed when we relate String-Nets with the Turaev-Viro State-sum and we need to consider the space of String-Nets that "avoid" vertices of some triangulation.

In Section 4 we describe the Turaev-Viro-Barret-Westbury state-sum invariant for PLCW 3-manifolds with boundary and we extend it to include a wider class of 3-bordims, namely pinched interval bordisms. This allows us, given an oriented closed surface Σ, to construct a set of locally commuting projectors for every embedded 2-disk $D^2 \subset \Sigma$ whose images, when intersected over a finite cover of Σ, are naturally isomorphic to $\mathcal{Z}_{\mathcal{C}}^{\mathrm{TV}}(\Sigma)$ i.e. the invariant vector space associated to the Turaev-Viro TQFT. This is an important step in understanding Turaev-Viro TQFT as a local TQFT.

In Section 5 we build the Levin-Wen model. We constrcut a Hilbert space for every oriented close manifold paired with a choice of embedded graph and we construct a set of local operators (also called the algebra of observables) for the model. From this set of data we build an exactly solvable Hamiltonian.

In Section 6 we bring together the three main construction of the thesis. We prove that the state-space $\mathcal{H}_{\mathrm{TV}}(\Sigma, \Delta)$ is naturally isomorphic to the space of String-Nets of the punctured surface $\Sigma - V(\Delta)$ and to the image of the vertex operators $\mathrm{Vertex}(\Sigma, \Delta^*)$. Starting from this result proceed to show that the ground-state subspace of the Levin-Wen Hamiltonian $\mathcal{Z}_{\mathcal{C}}^{\mathrm{LW}}(\Sigma, \Delta^*)$, the invariant vector space constructed from the Turaev-Viro TQFT $\mathcal{Z}_{\mathcal{C}}^{\mathrm{TV}}(\Sigma, \Delta)$ and the space of String-Nets $\mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma)$ are all isomorphic. Additionally, we define *Topological Hamiltonian Schemas* and we show that the Levin-Wen model is a Topological Hamiltonian schema corresponding to the Turaev-Viro/String-Net TQFT. In the end of the section we define *Topological Defect* for Hamiltonian schemas, as a formal and wider-ranging definition of what are more commonly referred to as *Anyons* or topological quasi-particles. Moreover, we point how to classify their types given an extended definition of the Levin-Wen Hamiltonian for open surfaces with open boundary conditions, in particular we equate the quest for classifying types of anyons with the quest for classifying the simple modules of the tube category of S^1.

In Section 7 we prove there is a *contravariant braided monoidal equivalence* between the category of modules of the tube category of S^1 and the Drinfel'd center $\mathcal{Z}(\mathcal{C})$. We construct this equivalence very carefully and provide equally detailed definitions of both adjunctions. We build a canonical restriction functor $\Psi\colon \mathcal{C} \longrightarrow \mathrm{Tube}_{\mathcal{C}}(S^1)$ and we work, as suggested in [9], not with the entire category of modules of the tube category of S^1 but with an equivalent subcategory of modules whose canonical restriction is equal to the co-Hom functor $\mathrm{Hom}(R, -)$ for some representing object R. Moreover, we construct, using the coend construction of Section 3, a monoidal strcuture for the category of modules and show that it also has a braided ribbon structure and show that the category of modules of the tube category is contravariantlu equivalent to the Drinfel'd center.

2 String-Nets

In this section, we introduce String-Nets. These are, in essence, to be understood as string-diagrams of a spherical fusion category $\mathcal{C}$ drawn, not only on planes and spheres, as we did in the last section, but on any oriented surface, which we consider to be piece-wise linear, along with all other manifolds in this thesis. Described differently they are algebraic gadgets, in so far as we will treat them as vector spaces, of labelled, often called coloured or decorated, cell-complexes that are properly embedded in an oriented surface and on which we impose the same graphical relations as the ones described in the last section. What we achieve is not necessarily a full-fledged TQFT. Instead we construct what is called a topless, (2+ϵ) or decapitated TQFT (see Walker's note for the conditions to extend a topless TQFT to (2+1)-TQFT [24, Ch.6]). These are TQFTs for which we do not specify a partition function for all 3-dimensional bordisms, but only those corresponding to the mapping cylinder of a diffeomorphism. Additionally, since our goal is understanding String-Nets as a local TQFT, we provide gluing rules for oriented surfaces with boundary and sketch a way to generalize it to surfaces with corners. It is also customary to call this "bottom" part of the TQFT a modular functor [2].

2.1 Labeled Graphs

Let Σ be an oriented surface which we will, unless stated otherwise, allow to have non-empty boundary and let $\mathcal{C}$ be a fixed spherical fusion category. Let us consider all properly embedded oriented finite graphs Γ in Σ, possibly with loops, which are allowed to have univalent vertices only on the boundary and so long this intersection is transverse to the boundary at that point. Additionally, edges of the graph cannot intersect the boundary.

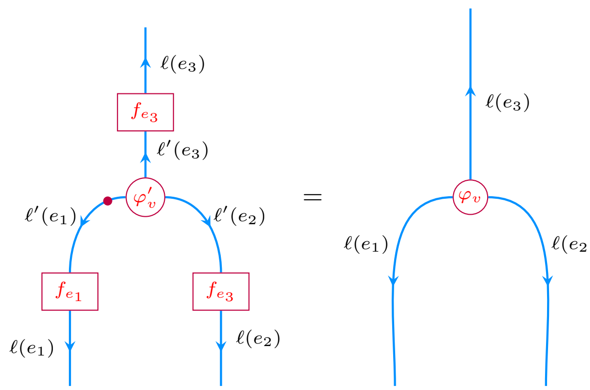
Definition 2.1. A $\mathcal{C}$-labeling ℓ of Γ, where Γ is a graph embedded in Σ as mentioned above, is the following collection of data:

- an object $\ell(e) \in \mathcal{C}$ for every oriented edge such that $\ell(\bar{e}) \simeq \ell(e)^*$, where the bar over e denotes orientation reversal;
- for all vertices v in the interior of Σ a choice of initial half-edge incident to v and a choice of morphism φ_v where:

$$\varphi_v \in \langle \ell(e_1), \dots, \ell(e_n) \rangle$$

  where the edges e_i are all the edges incident to v taken with outgoing direction and ordered counterclockwise starting at the initial half-edge.

We consider two labelings $\{\ell(e), \varphi_v\}, \{\ell'(e), \varphi'_v\}$ equivalent if there is an isomorphism $f_e\colon \ell'(e) \to \ell(e)$ that is compatible with the duality isomorphism $\ell(\bar{e}) \cong \ell(e)^*$ and should satisfy:



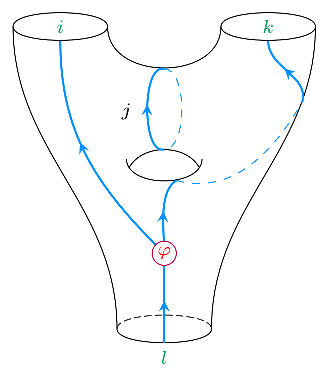
for all vertices and edges of Γ.

Every labeled graph comes naturally with information at the boundary. Every labeled graph meets $\partial\Sigma$ transversely at a finite set of points $B := \{b_1, \dots, b_n\} \subset \partial\Sigma$ and a collection of objects $\{v_b\}_{b\in B}$ where we establish the convention that $\{V_b\}_{b\in B}$ is defined with the orientation corresponding to the direction leaving Σ. We sum it up by saying Γ has $\{V_b\}$ as a boundary value.

We now define the embedding vector space for String-Nets by considering linear combinations of labelled graphs:

$$\mathcal{H}_{\mathcal{C}}\big(\Sigma; \{V_b\}\big) := \text{Span}_{\mathbb{C}}\{\mathcal{C}\text{-labelled graphs embedded in } \Sigma\}$$

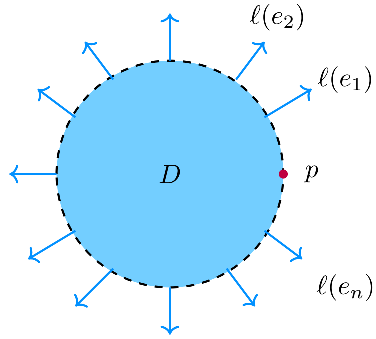
and provide a figure representing a basis vector in this space:



where $\varphi \in \langle l^*, i, k\rangle$.

In order to define the String-Net space, we must first introduce the extremely valuable notion of *evaluation*. Given the following data:

- an oriented surface Σ with a labelled embedded graph $\Gamma \in \mathcal{H}_{\mathcal{C}}\big(\Sigma; \{V_b\}\big)$;
- an embedded disk D with a marked point $p = \psi(1,0) \in \partial D$, for some embedding $\psi\colon D^2 \longrightarrow \Sigma$ of the disk into Σ

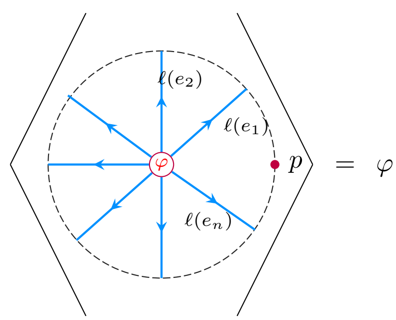


;

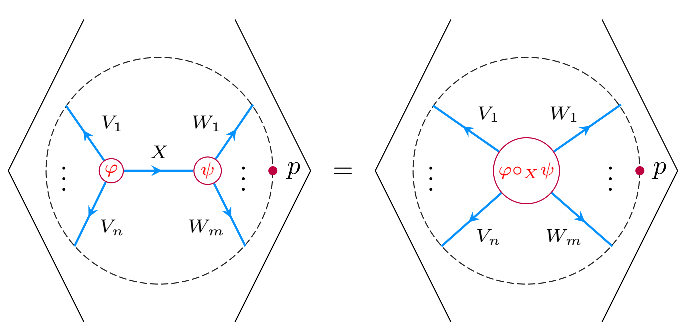
such that $\Gamma \cap \partial D$ contains no vertices and all edges of Γ are transverse to the boundary of the disk. We define the evaluation of the graph Γ at D to be the unique morphism $\langle \Gamma \rangle_D \in \langle \ell(e_1), \ldots, \ell(e_n) \rangle = \mathrm{Hom}_{\mathcal{C}}(1, \ell(e_1) \otimes \ldots \otimes \ell(e_n))$. The evaluation map allows us to locally i.e., in the image of a standard disk, enforce the graphical calculus rules, stemming from algebraic data defining the spherical fusion category $\mathcal{C}$. The following theorem provides some of the local relations satisfied by the evaluation map.

Theorem 2.2. *For a pair $\Gamma, \Gamma' \in \mathcal{H}_{\mathcal{C}}\big(\Sigma; \{B_b\}\big)$ and an embedded disk D. The following sequence of relations hold:*

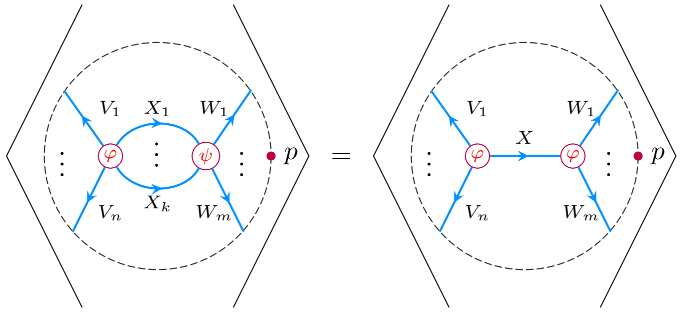
1. *If $\Gamma \cap D$ consists of a single vertex labeled by $\varphi \in \langle \ell(e_1), \ldots, \ell(e_n) \rangle$, then $\langle \Gamma \rangle_D = \varphi$*



2. *Isomorphic labelings in $\Gamma \cap D$ and $\Gamma' \cap D$ imply that $\langle \Gamma \rangle_D = \langle \Gamma' \rangle_D$.*
3. *If there is an isotopy, relative to the boundary, of D connecting $\Gamma \cap D$ and $\Gamma' \cap D$ then $\langle \Gamma \rangle_D = \langle \Gamma' \rangle_D$.*
4. *Rotating choices of the inital half-edge preserve the evaluation.*
5. *Vertex merge move:*



6. *Edge merge move:*



where $X = X_1 \otimes \ldots \otimes X_k$

7. *Some additional relations adding to do with compatibility of direct sum decomposition of objects and morphisms.*

We are now in a position to put forward a definition of the vector space that String-Net TQFTs associate to surfaces.

Definition 2.3. (String-Net Space) Let Σ be an oriented surface, possibly with empty boundary, an embedded disk $D \subset \Sigma$ and a fixed choice $(B, \{V_b\}_{b \in B})$ of boundary condition. We deem a labeled graph $\Gamma \in \mathcal{H}_{\mathcal{C}}\big(\Sigma; \{V_b\}\big)$ to be null, when evaluated at D, if:

$$\langle \Gamma \rangle_D = 0$$

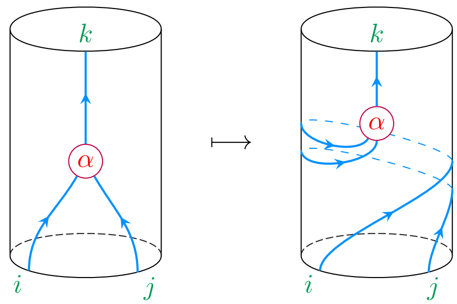
then we define the String-Net space as the following quotient:

$$\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\big(\Sigma; \{V_b\}\big) := \mathcal{H}_{\mathcal{C}}\big(\Sigma; \{V_b\}\big) / \sim$$

where the $\Gamma_1 \sim \Gamma_2$ if and only if $\langle \Gamma_1 - \Gamma_2 \rangle_D$ is a null graph for all possible embedded disks $D \subset \Sigma$. We denote the equivalence class of Γ as $\langle \Gamma \rangle$.

We defined local relations for labeled graphs in embedded disks and a fortiori imposed a global relation by enforcing the local relations in every open ball of Σ. Moreover, the following list of local relations also hold as global relations:

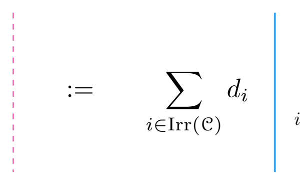
- global istopies;
- isomorphic labellings;
- vertex and edge direct sum decomposition;
- the String-Nets act as type of "exponentiation" by mapping disjoint unions of surfaces to the tensor product of vector spaces:
$$\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\Big(\Sigma_1 \sqcup \Sigma_2, \{V_b\}\Big) \cong \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\Big(\Sigma_1, \{V_{b_1}\}\Big) \otimes \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\Big(\Sigma_2, \{V_{b_2}\}\Big)$$
a good footprint of the quantum nature of the construction.
- if $f \colon \Sigma_1 \longrightarrow \Sigma_2$ is a diffeomorphism, then there is a natural isomorphism:
$$f_{\#} \colon \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\big(\Sigma_1, \{V_b\}\big) \longrightarrow \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\big(\Sigma_2, \{V_b\}\big) \tag{2.1}$$
since String-Nets are properly embedded submanifolds As an example of this behaviour we can consider the Dehn twist of the cylinder:
$$\theta \colon S^1 \times I \longrightarrow S^1 \times I$$
its action on the space of String-Nets is given by rotating the String-Net along the diffeomorphism. The picture below exemplifies this process for an arbitrary String-Net:



String-Nets have been a part of the mathematical physics literature for the last 15 years, often under distinct names, like picture TQFTs, string-diagram TQFTs, diagram TQFTs or Levin-Wen models in the physics literature. They have recently featured more prominently in connections with the Turaev-Viro State-Sum [12]. In particular they are often stated as being the continuum theory of Turaev-Viro, as they form an equivalent TQFT without the need to reference a fixed cellular decomposition. On the other hand they provide an important link in interpreting the connection between the Levin-Wen models and the Turaev-Viro state-sum.

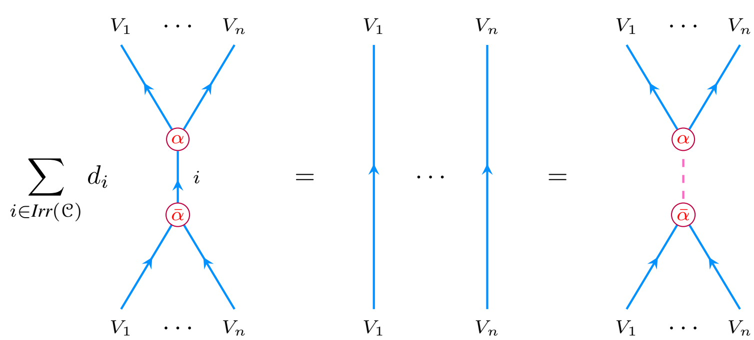
Let us write down some useful local relations of which we will consistently make use 2.2:

Proposition 2.4. *We begin by defining the following graphical notation:*

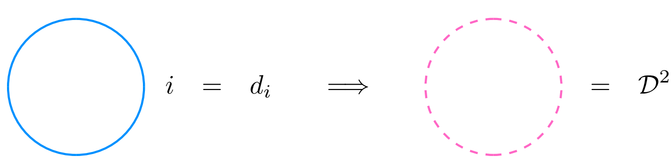


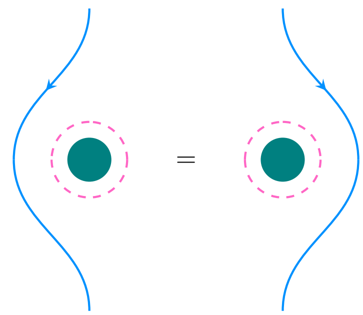
then we have the following relations:

- *(Resolution)*



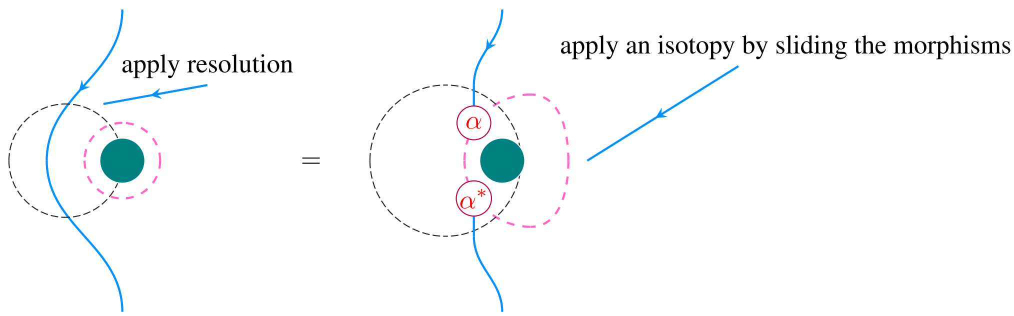
- *(Loop Contraction)*



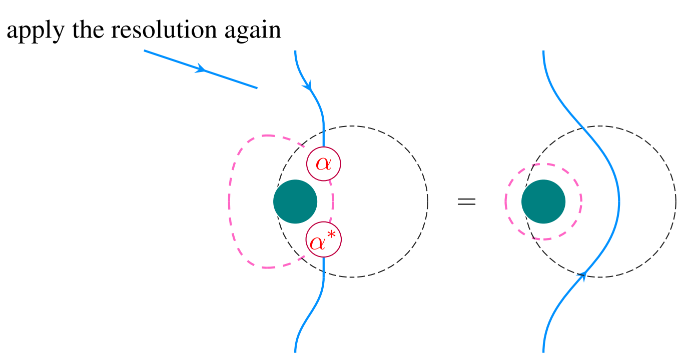
- *(Cloaking Relation)* 

the dark green region is meant to represent a puncture, any "hole", or any region possibly with String-Nets and boundary in the surface.

Proof. Let us look at a proof of the only non-obvious of these relations, the cloaking relation:

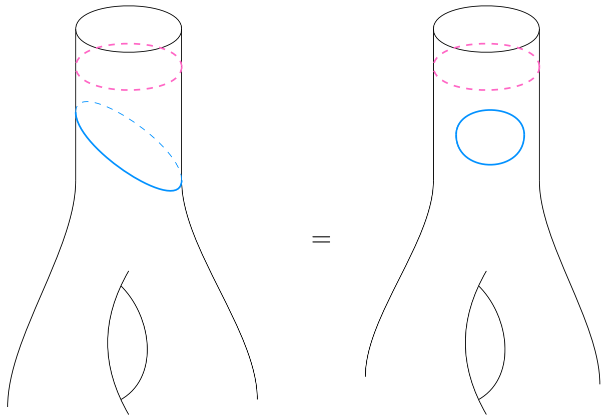


and after effectively sliding the α and α^* morphisms from the left to the right-hand side of the picture we get the following equality:



□

The power of the cloaking relation, which leads us to define  as the cloaking element, is that we can use it to "hide" possible topological obstructions like boundaries or punctures in the surface. By hiding we mean that they become undetectable to String-Nets allowing them to move past them. As an example:

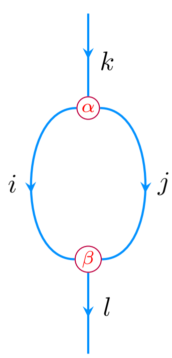


we see how the string in blue, which we chose, for generality purposes, to remain unlabeled, can completely "avoid" the puncture so long as the latter is encircled by the cloaking element. This is a powerful tool in the behavioural study of String-Net models.

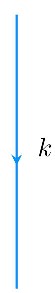
2.2 Fusion Rules as Local Relations

In the previous subsection we defined the space of String-Nets on an oriented surface Σ by considering a quotient of the vector space of $\mathcal{C}$-labeled graphs on Σ. This quotient is constructed by modding out all labelled graphs that differed by what we have been calling a local relation. A local relation between a pair of labelled graphs can be seen as an equation involving two labelled graphs that differ at most inside a disk embedded in the interior of Σ. We gave some examples of local relations in Theorem 2.2 and Theorem **??**. We will look at these in greater detail and consider the opportunity to include a definition of F-symbols for spherical fusion categories.

From this section onward we will employ a change in the normalization of the dual basis for $\mathrm{Hom}_{\mathcal{C}}(V_i \otimes V_j, V_k)$. In particular, we will choose a dual basis or which the pairing has the following coefficients:



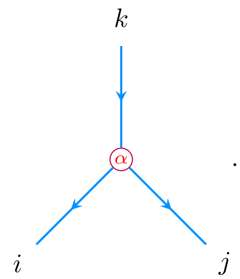
$$= \quad \delta_{\beta,\alpha^*}\delta_{k,l} \cdot \sqrt{\frac{d_i d_j}{d_k}} \tag{2.2}$$



Moreover, we will assume that the categories we are working with are unitary, meaning that we will assume that all of our structural morphism data is unitary. For a discussion on the choice of this normalization conditionm and why its a direct consequence of unitarity, we recommend Wen and Lan's discussion in [15]. To begin the discussion let us first introduce a piece of notation:

$$N_{ij}^k := \mathrm{Hom}_{\mathcal{C}}(V_k, V_i \otimes V_j) = \langle i, j, k^* \rangle. \tag{2.3}$$

These are often called fusion spaces in the physics literature and they form the Hilbert space to which the diagram below belongs to:



Recall that, in a fusion category all objects are isomorphic to a finite direct sum of simple objects. Hence, for any $V_i, V_j, V_k \in \mathrm{Irr}(\mathcal{C})$ we have the following isomorphism:

$$V_i \otimes V_j \cong \bigoplus_{n \in \mathrm{Irr}(\mathcal{C})} N_{ij}^n \cdot V_n \tag{2.4}$$

Since fusion categories are, in particular, monoidal categories, for all triple of objects there exists a natural isomorphism:

$$a_{V_i,V_j,V_k} \colon (V_i \otimes V_j) \otimes V_k \longrightarrow V_i \otimes (V_j \otimes V_k)$$

which impose additional constraints on these fusion spaces N_{ij}^k. The following calculation elucidates this point a little further. Notice that, since the associativity isomorphism is a natural one, we have the following constraints for the fusion spaces:

$$\mathrm{Hom}_{\mathcal{C}}\left(V_l, (V_i \otimes V_j) \otimes V_k\right) \cong \mathrm{Hom}_{\mathcal{C}}\left(V_l, V_i \otimes (V_j \otimes V_k)\right) \tag{2.5}$$

$$\mathrm{Hom}_{\mathcal{C}}\left(V_l, \bigoplus_{n \in \mathrm{Irr}(\mathcal{C})} N_{ij}^n \cdot (V_n \otimes V_k)\right) \cong \mathrm{Hom}_{\mathcal{C}}\left(V_l, \bigoplus_{m \in \mathrm{Irr}(\mathcal{C})} N_{jk}^m \cdot (V_i \otimes V_m)\right) \tag{2.6}$$

$$\mathrm{Hom}_{\mathcal{C}}\left(V_l, \bigoplus_{n,p \in \mathrm{Irr}(\mathcal{C})} (N_{ij}^n \otimes N_{nk}^p) \cdot V_p\right) \cong \mathrm{Hom}_{\mathcal{C}}\left(V_l, \bigoplus_{m,q \in \mathrm{Irr}(\mathcal{C})} (N_{jk}^m \otimes N_{im}^q) \cdot V_q\right) \tag{2.7}$$

$$\bigoplus_{n \in \mathrm{Irr}(\mathcal{C})} (N_{ij}^n \otimes N_{nk}^l) \cdot V_l \cong \bigoplus_{m \in \mathrm{Irr}(\mathcal{C})} (N_{jk}^m \otimes N_{im}^l) \cdot V_l. \tag{2.8}$$

Where the tensor products between fusion spaces correspond to the usual tensor product over complex vector space, which is symmetric i.e. $V \otimes W \cong W \otimes V$, and so we can rewrite equation 2.8 as follows:

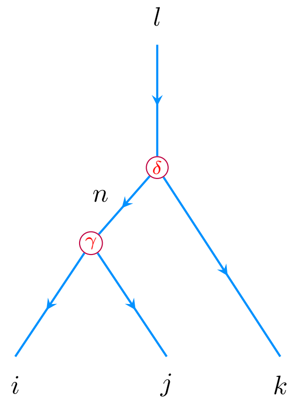
$$\bigoplus_{n \in \mathrm{Irr}(\mathcal{C})} (N_{nk}^l \otimes N_{ij}^n) \cdot V_l. \cong \bigoplus_{m \in \mathrm{Irr}(\mathcal{C})} (N_{im}^l \otimes N_{jk}^m) \cdot V_l \tag{2.9}$$

Granting us an isomorphism:

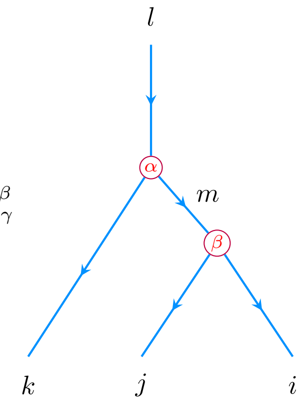
$$F_l^{ijk} \colon \bigoplus_{n \in \mathrm{Irr}(\mathcal{C})} N_{nk}^l \otimes N_{ij}^n \longrightarrow \bigoplus_{m \in \mathrm{Irr}(\mathcal{C})} N_{im}^l \otimes N_{jk}^m, \tag{2.10}$$

the components of which we will call F-symbols. and whose graphical . The graphical interpretation for this isomorphism is given by the next definition.

Definition 2.5. Let $V_i, V_j, V_n, V_l \in \mathrm{Irr}(\mathcal{C})$, then we have the following:



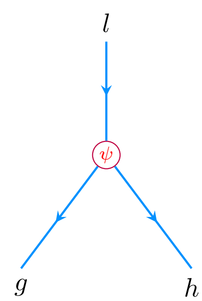
$$= \sum_{\alpha,\beta} \sum_{m \in \mathrm{Irr}(\mathcal{C})} (F_l^{ijk})^{\alpha n \beta}_{\delta m \gamma}$$



where δ is a basis element of N^l_{nk} and γ a basis element for N^n_{ij}, α runs through a basis of N^l_{im} and β runs through a basis of N^m_{jk}. The scalars $(F_l^{ijk})^{\alpha n \beta}_{\delta m \gamma}$ are called the F-**symbols** of $\mathcal{C}$. Moreover, if besides being an isomorphism F_l^{ijk} is also a unitary operator, then we say that $\mathcal{C}$ is a **unitary**.

When doing physics, unitarity of $\mathcal{C}$ is required. As such, and since we will be interested in working with physical models in subsequent sections, we must preempt a small discussion on the normalization coefficients of the dual basis.

Example 2.6. If we consider the example of $\mathcal{C} = \mathbb{C} - \mathrm{Vect}[G]$, then we have that:



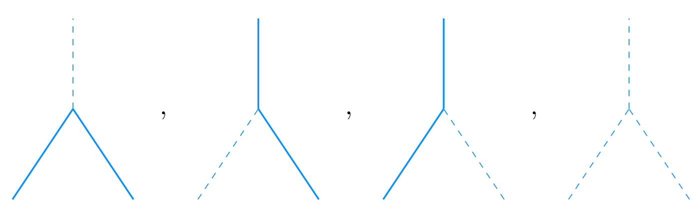
So that $\psi \in \mathrm{Hom}_{\mathcal{C}}(l, g \otimes h) = \mathrm{Hom}_{\mathcal{C}}(l, gh)$ and hence $\psi = 0$ if $l \neq gh$ and $\psi = \lambda \cdot \mathrm{id}_{gh}$ otherwise. In which case

$$(F_l^{ghk})^n_m = \delta_{l,ghk}\delta_{m,gh}\delta_{n,hk}$$

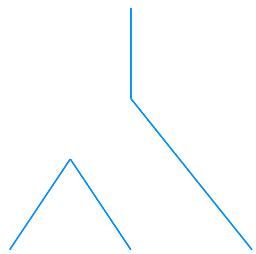
Example 2.7. Notice that using the example above we get that for $G = \mathbb{Z}_2$, we get the following simple fusion rules:

$$\begin{aligned} 1 \otimes 1 &= 0 \\ 1 \otimes 0 &= 1 \\ 0 \otimes 1 &= 1 \\ 0 \otimes 0 &= 0 \end{aligned}$$

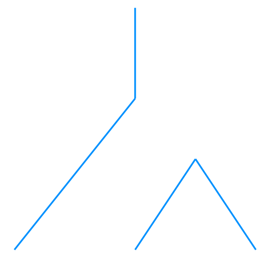
which means that the only non-zero fusion spaces are given by the following sequence:



where we represent 1 as the solid line and 0 as the dashed line. If we follow convention and use "transparent" strands instead of dashed ones, then one of the local relations afforded by the F-symbols is the following :



$=$



where we made use the fact that $(F_1^{111})_0^0 = 1$. Here we did not make use of any red marker in our diagrams because all Frobenius-Schur indicators are trivial.

Example 2.8. Let us consider the famous Ising category. The Ising category is generated by three simple objects, which we will denote by $1, \sigma, \psi$, where 1 naturally stands for the monoidal unit. They have the following non-trivial fusion rules:

$$\sigma \otimes \sigma \cong 1 \oplus \psi \tag{2.11}$$
$$\sigma \otimes \psi \cong \psi \otimes \sigma \cong \sigma \tag{2.12}$$
$$\psi \otimes \psi \cong 1 \tag{2.13}$$

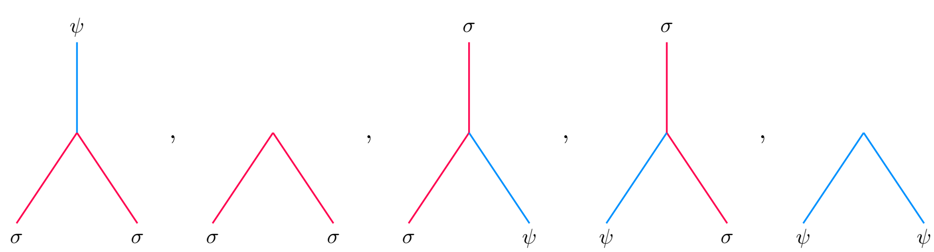
Additionally, the only non-trivial F-symbols are given given by:

$$F_\sigma^{\sigma\sigma\sigma} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \tag{2.14}$$
$$(F_\psi^{\sigma\psi\sigma})_\sigma^\sigma = (F_\sigma^{\psi\sigma\psi})_\sigma^\sigma = -1 \tag{2.15}$$

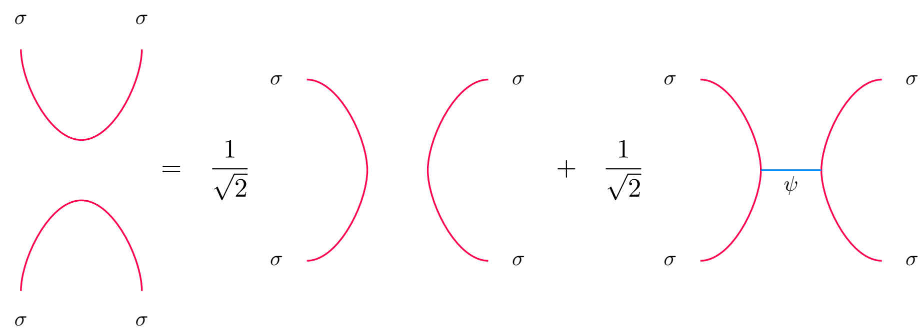
To get a better feel for these relations we choose to represent the simple objects as follows:

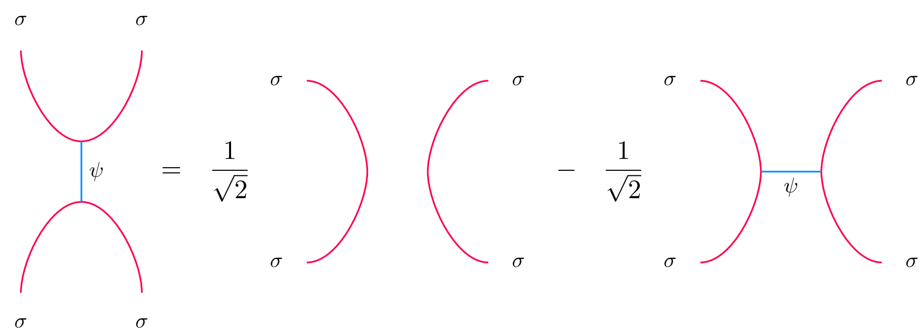
$\sigma =$, $\psi =$

with the unit given by the transparent strand. With this choice of notation the fusion rules are represented by the followings set of morphisms:



in which case we can represent some of the F-moves as follows: We chose to curve the strands and isotope them to exhibit the F-moves in a slighly more natural and fluid fashion within the context of String-Nets.

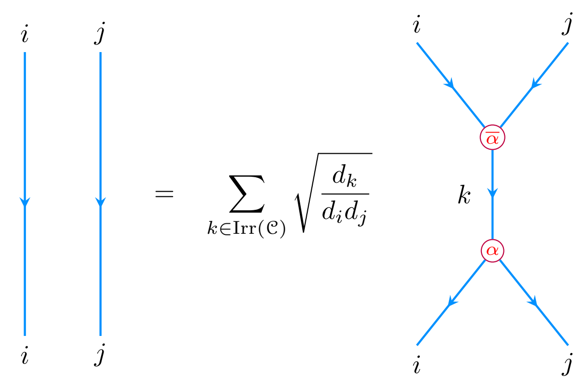




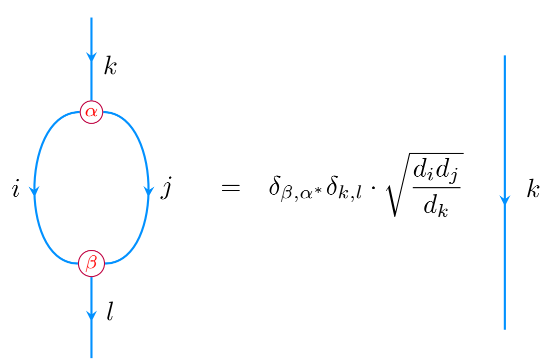
These form a great set of examples for non-trivial and non-generic local relations. A concrete example of this of these rules directly applied to a String-Net example is given

Proposition 2.9. *The following list contains a handful of useful local relations for String-Nets in an arbitrary spherical fusion category $\mathcal{C}$. We make use of our newly defined normalization convention for basis vectors of the fusion spaces:*

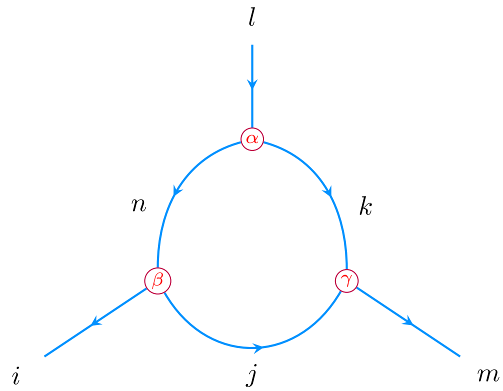
- *(Resolution of Identity)*



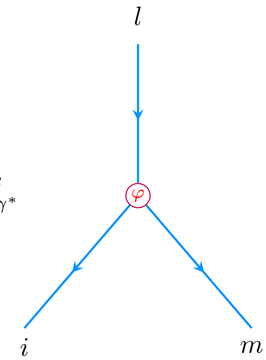
- *(Bubble-Removal)*



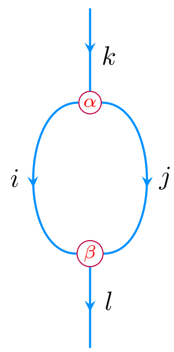
- *(Sputnik-Killer)*



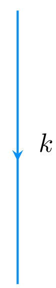
$$= \sum_{\varphi} \sqrt{\frac{d_j d_k}{d_m}} (F_l^{ijk})^{\alpha n \beta}_{\varphi m \gamma^*}$$



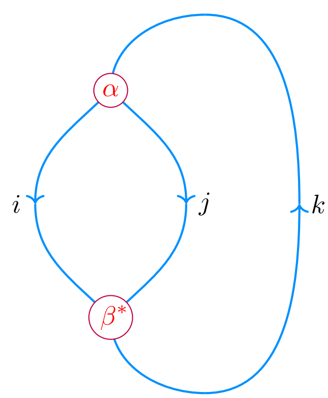
Proof. For the bubble removal we have that, from Schur's lemma:



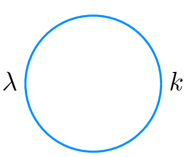
$$= \quad \delta_{k,l} \cdot \lambda$$



and taking the trace from both sides of the equation we get:

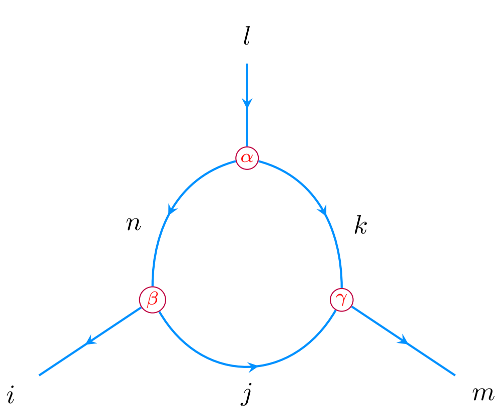


$$= \delta_{k,l} \cdot \lambda$$

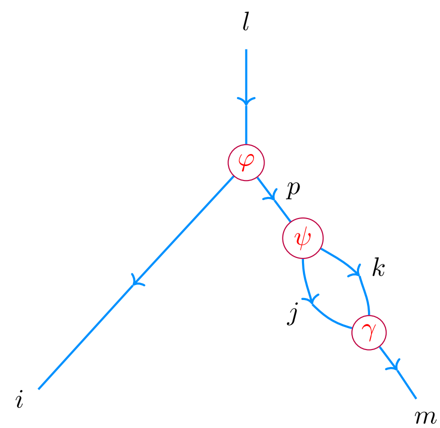


which can be rewritten, using as:

$$\sqrt{d_i d_j d_k} \cdot \langle \beta, \alpha \rangle = \lambda \cdot d_k \iff \lambda = \delta_{\beta,\alpha^*} \sqrt{\frac{d_i d_j}{d_k}}$$

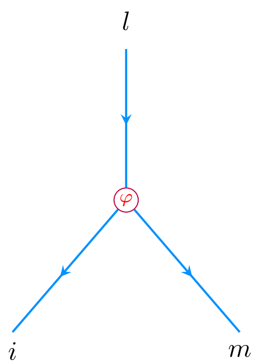


$$= \sum_{\varphi,\psi} \sum_{p\in \mathrm{Irr}(\mathcal{C})} (F_l^{ijk})_{\alpha n \beta}^{\varphi p \psi}$$

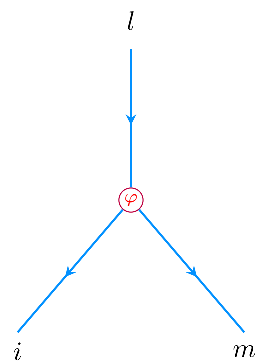


Where we apply an F-move on the right-hand side of the diagram, including coupons β and γ. Next, apply the bubble removal to the ψ and γ pair. To conclude all we have to is adjust the summation indices to match with the Kronecker deltas that emerged from the bubble removal.

$$\sum_{\varphi,\psi} \sum_{p\in \mathrm{Irr}(\mathcal{C})} (F_l^{ijk})_{\alpha n \beta}^{\varphi p \psi} \delta_{\gamma,\psi^*} \delta_{m,p} \sqrt{\frac{d_j d_k}{d_m}}$$

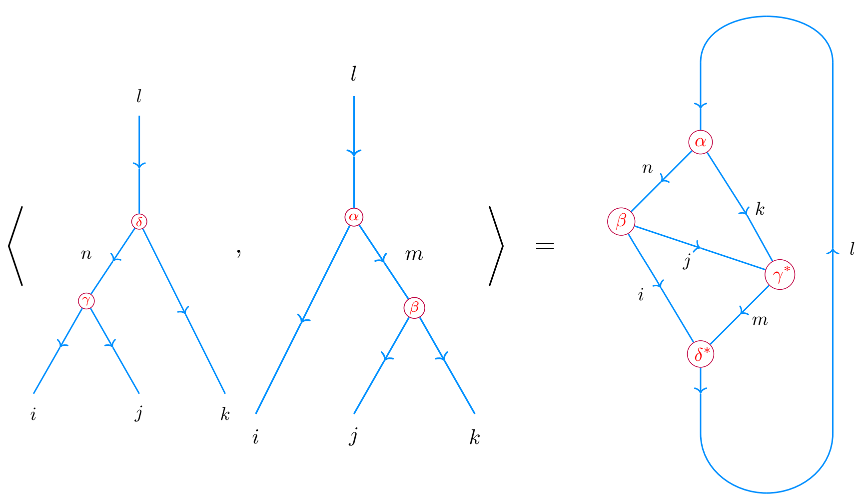


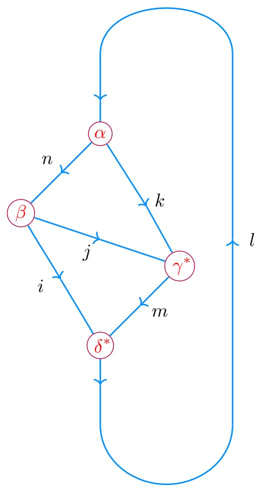
$$= \sum_{\varphi} (F_l^{ijk})_{\alpha n \beta}^{\varphi m \gamma^*} \sqrt{\frac{d_j d_k}{d_m}}$$



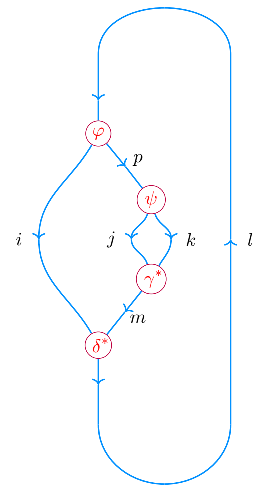
concluding the proof. □

We have defined F-symbols and we have attested how useful they are for manipulating String-Nets, but we still have not made explicit how to compute them. Since their values directly show up in the Turaev-Viro state-sum, we show here their direct computation, to be used in subsequent sections. The F-symbols are the components of the the F_l^{ijk} operator, so we can use the pairing

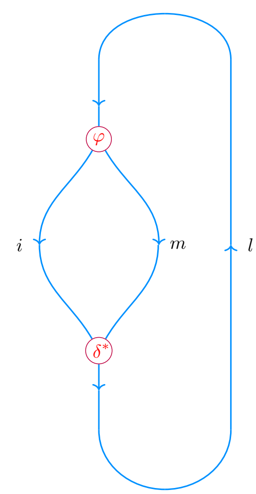




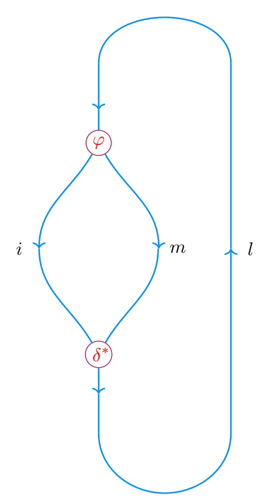
$$= \quad \sum_{\varphi,\psi} \sum_{p\in \mathrm{Irr}(\mathcal{C})} (F_l^{ijk})^{\alpha n\beta}_{\varphi p\psi}$$



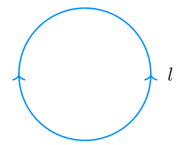
$$= \quad \sum_{\varphi,\psi} \sum_{p\in \mathrm{Irr}(\mathcal{C})} \delta_{\psi,\gamma^*}\delta_{p,m}\sqrt{\frac{d_j d_k}{d_m}} \cdot (F_l^{ijk})^{\alpha n\beta}_{\varphi p\psi}$$



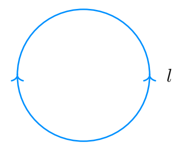
$$= \quad \sum_{\varphi} \sqrt{\frac{d_j d_k}{d_m}} \cdot (F_l^{ijk})^{\alpha n\beta}_{\varphi m\gamma^*}$$



$$= \quad \sum_{\varphi} \delta_{\varphi,\delta^*} \sqrt{\frac{d_i d_m}{d_l}} \sqrt{\frac{d_j d_k}{d_m}} \cdot (F_l^{ijk})^{\alpha n \beta}_{\varphi m \gamma^*}$$



$$= \quad \sqrt{\frac{d_i d_m}{d_l}} \sqrt{\frac{d_j d_k}{d_m}} \cdot (F_l^{ijk})^{\alpha n \beta}_{\delta^* m \gamma^*}$$



$$= \quad \sqrt{\frac{d_i d_m}{d_l}} \sqrt{\frac{d_j d_k}{d_m}} \cdot (F_l^{ijk})^{\alpha n \beta}_{\delta^* m \gamma^*} \cdot d_l$$

$$= \quad \sqrt{d_i d_j d_k d_l} \cdot (F_l^{ijk})^{\alpha n \beta}_{\delta^* m \gamma^*}$$

2.3 Tube Category

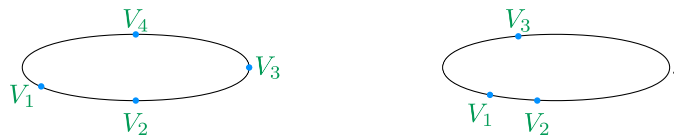
In this subsection we intend to make the case, in a similar manner to the second section, for why the string-net space holds all the properties needed for a modular functor. In order to do this we make great use of the all-important notion of tube category. Tube categories serve as natural categorified versions of the tube algebra, an algebraic gadget that has been part of the literature of both subfactor theory and topological field theory for the last 25 years. In the recent past it has been put to extensive use as tool to study the excitation spectrum for a variety of topolgical lattice models. These topological lattice models, in particular the Levin-Wen models in the $(2+1)$-dimension case, are thought to classify all non-chiral topological phases of matter in $(2+1)$-dimensions and whose physics are governed by the mathematics of unitary TQFTs. What this means is that:

- the ground-state subspace of these lattice models, on a given closed oriented surface Σ, is described by the invariant subspace associated to it by some unitary TQFT to that same surface;
- the time-evolution of the lattice model is described by the unitary linear operator associated to the triangulated 3-manifold M, for which $\Sigma \in \partial M$, traced by Σ over time.

It turns out that to study possible excitations of these models, the language of categories, in particular the representation theory of categories, becomes indispensable. The string-net picture we have been developing over the last section is meant, in no ambiguous terms, to construct an alternative method of understanding the Levin-Wen lattice models, whose unitary TQFT, as we will check later on, corresponds to Turaev-Viro, by building a TQFT that is equivalent to Turaev-Viro although less reliant on the "brute-force" state-sum nature in which the latter is typically presented. Part of the appeal of this shift in perspective comes from the notion of extended TQFTs.

Definition 2.10. (Tube Category) Let S be a closed oriented 1-manifold i.e., a collection of disjoint circles. We define the following category $\mathrm{Tube}_{\mathcal{C}}(S)$:

- Objects are given by string-net boundary conditions $\{V_b\}_{b\in B}$. Recall that $B \subset S$ is a finite subset of S and V_b is an object of $\mathcal{C}$. Examples are given by the following pictures

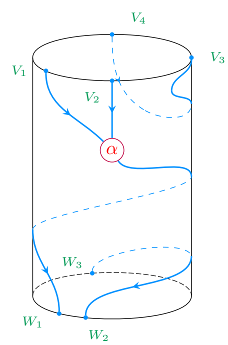


;

- Morphisms are given by string-nets defined on $S \times I$, where $I = [-1, 1]$.

$$\mathrm{Hom}_{\mathrm{Tube}_{\mathcal{C}}(S)}\left(\{V_b\}, \{W_{b'}\}\right) = \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\left(S \times I; \{V_b\}, \{W_{b'}\}^{\vee}\right) \tag{2.16}$$

where we define $\{V_b\}^{\vee} := \{V_b^*\}$. Composition is given gluing the cylinders along S, which is well-defined since $(S \times I) \cup_S (S \times I) \cong S \times I$ and we have ensured that this a well-defined process when we use $\{V_b\}^{\vee}$, rather than $\{V_b\}$ on the left boundary condition.



There is a natural identification of the tube category for $-S$ with the opposite of the category of S:

$$\mathrm{Tube}_{\mathcal{C}}(-S) \cong \mathrm{Tube}_{\mathcal{C}}(S)^{\mathrm{op}} \tag{2.17}$$

It is proved as a consequence of the fact that $-S \times I$ can be identified with a vertical reflection of $S \times I$. Additionallty, we have the important property of the tube category:

$$\mathrm{Tube}_{\mathcal{C}}(S_1 \sqcup S_2) \cong \mathrm{Tube}_{\mathcal{C}}(S_1) \times \mathrm{Tube}_{\mathcal{C}}(S_1) \tag{2.18}$$

Theorem 2.11. *Let Σ be an oriented surface with possibly non-empty boundary. We have that the string-net space $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma, -)$ can be recast as linear contravariant functor from the tube category associated to $\partial\Sigma$ to the category* Vect *of finite-dimensional complex vector spaces. The functor is defined in the following natural manner on objects :*

$$\begin{aligned} \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma, -)\colon\ \mathrm{Tube}_{\mathcal{C}}(\partial\Sigma) &\longrightarrow \mathrm{Vect} \\ \{V_b\} &\longmapsto \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\left(\Sigma, \{V_b\}\right) \end{aligned}$$

for morphisms, it is defined by gluing $\Gamma \in \mathrm{Hom}_{\mathrm{Tube}_{\mathcal{C}}(\partial\Sigma)}\left(\{V_b\}, \{W_{b'}\}\right)$ i.e., a string-net on $\partial\Sigma \times I$ – along the boundary of Σ. The corresponding operator is defined as follows:

$$\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\left(\Sigma, \boxed{\Gamma}\right) : \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\left(\Sigma, \{W_{b'}\}\right) \longrightarrow \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\left(\Sigma, \{V_b\}\right) \tag{2.19}$$

$$\xi \longmapsto \Gamma \cdot \xi \tag{2.20}$$

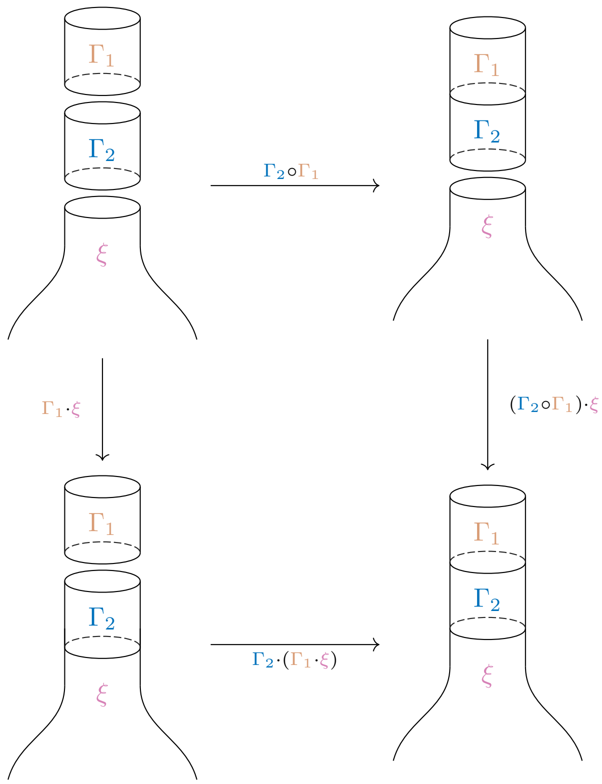
This is a well-defined linear map since there is a natural isomorphism:

$$\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\left(\Sigma \cup_{\partial\Sigma} (\partial\Sigma \times I); -\right) \cong \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma, -) \tag{2.21}$$

Proof. In order to make sure we have indeed defined a functor, we need to check the following points:

- (Functoriality) $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma, \Gamma_2 \circ \Gamma_1) = \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma, \Gamma_1) \circ \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma, \Gamma_2)$
- (Identity axiom) $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma, \mathrm{id}_{\{V_b\}}) = \mathrm{id}_{\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma, \{V_b\})}$

Covariant functoriality comes from the observation that the following diagram is commutative:



To check the identity axiom we note that the identity of $\{V_b\}$ is the diagram with identity strands connecting all the objects $\mathrm{id} \in \mathrm{Hom}_{\mathrm{Tube}_{\mathcal{C}}(\mathcal{C})}\big(\{V_b\}, \{V_b\}\big)$. As a consequence, the linear map associated to it acts on $\xi \in \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma, \{V_b\})$ by "doing nothing", in other words it leaves $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\Big(\Sigma, \{V_b\}\Big)$ unchanged i.e., it is the identity map. $\square$

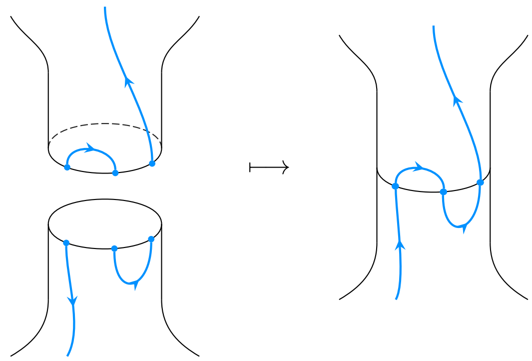
Corollary 2.12. *The string-net functor of an oriented surface* Σ*, constructed in the theorem above, is an object in the representation catgeory of* $\mathrm{Tube}_{\mathcal{C}}(\partial\Sigma)$*. In simplified, category-theoretic language,* $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma, -) \in \mathrm{Rep}(\mathrm{Tube}_{\mathcal{C}}\big(\partial\Sigma)\big)$.

Our next focus is the study of how this representation behaves under gluing. In order for the string-net model to be a genuine extended TQFT, we need to ensure that the vector space $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{V_b\})$ is recovered if we "chop it" into simpler pieces. In fact, this is one of the hallmark achievements of extended TQFTs, the possibility of constructing a complicated invariant out of data on simpler manifolds so long as there exists a well-defined gluing operation. That is precisely the next step we take. To do so, we start by introducing the gluing map.

Definition 2.13. Let Σ be an oriented, not necessarily connected surface with possibly non-empty boundary. Also, let us assume for the sake of our argument that there is a pair of boundary components $-S \sqcup S \in\subseteq \Sigma$. Then we have a linear operator:

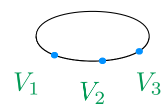
$$\mathrm{glue}_{\{V_b\}} \colon \mathcal{H}_{\mathcal{C}}(\Sigma; \{V_b\}, \{V_b\}^{\vee}) \longrightarrow \mathcal{H}_{\mathcal{C}}(\hat{\Sigma}_S)$$

Which we define by identifying states in $\mathcal{H}_{\mathcal{C}}(\Sigma; \{V_b\}, \{V_b\}^\vee)$ with states in $\mathcal{H}_{\mathcal{C}}(\hat{\Sigma}_S)$, as the following picture suggests:



where:

$$\{V_b\} \quad :=$$



Here, and throughout the rest of the following discussion, we abuse notation when using $\mathcal{H}_{\mathcal{C}}(\Sigma; \{V_b\})$ and $\mathcal{H}_{\mathcal{C}}(\hat{\Sigma}_S)$, in full rigour we should have included a boundary condition for all other boundary components, instead we chose to highlight exclusively the boundary conditions in $-S$ and S respectively. The gluing map descends to the quotient since gluing commutes with the application of local relations:

$$\mathrm{glue}_{\{V_b\}} \colon \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{V_b\}^\vee, \{V_b\}) \longrightarrow \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S) \tag{2.22}$$

We, once again, abuse notation by not changing the name of the map. Given the fact that we will exclusively work with the latter there is really no cause for concern. We end the definition by giving a partial answer to how the vector space $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S)$ can be computed via the knowledge of $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma)$. Part of the answer has to with the map:

$$\mathrm{glue} \colon \bigoplus_{\{V_b\}} \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{V_b\}^\vee, \{V_b\}) \longrightarrow \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S) \tag{2.23}$$

defined by the commutativity of the following diagram:

$$\begin{tikzcd}
\bigoplus_{\{V_b\}} \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{V_b\}^\vee, \{V_b\}) \arrow[r, "\pi_{\{V_b\}}"] \arrow[dr, dashed, "\mathrm{glue}"'] & \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{V_b\}^\vee, \{V_b\}) \arrow[d, "\mathrm{glue}_{\{V_b\}}"] \\
 & \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S)
\end{tikzcd}$$

i.e., $\mathrm{glue} = \mathrm{glue}_{\{V_b\}} \circ \pi_{\{V_b\}}$.

To understand in what way this global gluing map might help us capture further properties of the string-net model let us first examine its properties as a linear map.

Lemma 2.14. *Let Σ and S be as in the previous discussion. Then we have that* glue *is a surjective linear map.*

Proof. Consider an arbitrary string-net $\xi \in \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S)$, then without changing its equivalence class, we can pick, through a multitude of representatives, via isotopy relations or any other sequence of local moves, a string-net ω that is transverse to the gluing region and whose intersection with it is described by $\{V_b\}$):

- $\omega|_S = \{V_b\}$;
- $\mathrm{glue}(\omega) \sim \xi$

since ξ was arbitrary, the proof is concluded. $\square$

From the proof of the above lemma, we can also easily infer that glue is definitely non-injective, in other words $\ker(\mathrm{glue}) \neq \{0\}$. As such, understanding the following exact sequence:

$$0 \longrightarrow \ker(\mathrm{glue}) \hookrightarrow \bigoplus_{\{V_b\}} \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{V_b\}^\vee, \{V_b\}) \xrightarrow{\mathrm{glue}} \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S) \longrightarrow 0 \tag{2.24}$$

might provide the tools to build $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma})$ from $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma, -)$.

Theorem 2.15. *Let Σ and S be as above. Then we have that, for all $\{V_b\} \in \mathrm{Tube}_{\mathcal{C}}(S)$, there exists a linear map:*

$$\mathrm{glue}_{\{V_b\}} \colon \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{V_b\}^\vee, \{V_b\}) \longrightarrow \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S) \tag{2.25}$$

such that, for every morphism $\Gamma \colon \{V_b\} \longrightarrow \{W_c\}$ the following diagram commutes:

$$\begin{tikzcd}
 & \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{V_b\}^\vee, \{V_b\}) \arrow[dr, "\mathrm{glue}_{\{V_b\}}"] & \\
\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{V_b\}^\vee, \{W_c\}) \arrow[ur, "\mathrm{id}_{\{V_b\}} \times \Gamma"] \arrow[dr, "\Gamma^{\mathrm{op}} \times \mathrm{id}_{\{W_c\}^\vee}"'] & & \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S) \\
 & \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{W_c\}^\vee, \{W_c\}) \arrow[ur, "\mathrm{glue}_{\{W_c\}}"'] &
\end{tikzcd} \tag{2.26}$$

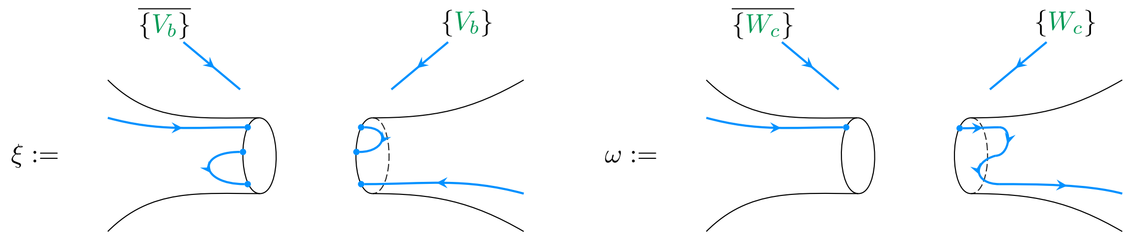
Before closing the theorem we provide additional clarification on notation. Note that, in full rigour, every time we work with string-nets on Σ we should really specify all boundary conditions in $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; -, \dots, -)$, since we are working exclusively with the pair $-S, S$ we always assume a particular boundary condition for all other components and keep them omitted. Additionally, we also need to clarify the notation used on the top left and bottom left arrows:

$$\mathrm{id}_{\{V_b\}} \times \Gamma := \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; (\Gamma \times \mathrm{id}_{\{V_b\}}) \colon \{W_c\}^\vee \times \{V_b\} \to \{V_b\}^\vee \times \{V_b\})$$

$$\Gamma^{\mathrm{op}} \times \mathrm{id}_{\{W_c\}^\vee} := \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; (\Gamma \times \mathrm{id}_{\{V_b\}}) \colon \{W_c\}^\vee \times \{V_b\} \to \{W_c\}^\vee \times \{W_C\})$$

Proof. The core of the proof comes from the fact that shifting a collar across the gluing submanifold only changes the string-net by an isotopy. $\square$

To get a better understanding for the topological flavour of the last proof and simultaneously garner extra intuition for why it is of importance to our study, let us look at the the following pair of pictures:



on the left we have a string-net in $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{V_b\}^\vee, \{V_b\})$ and on the right a string-net in $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{W_c\}^\vee, \{W_c\})$. We have, by an abuse of notation, that:

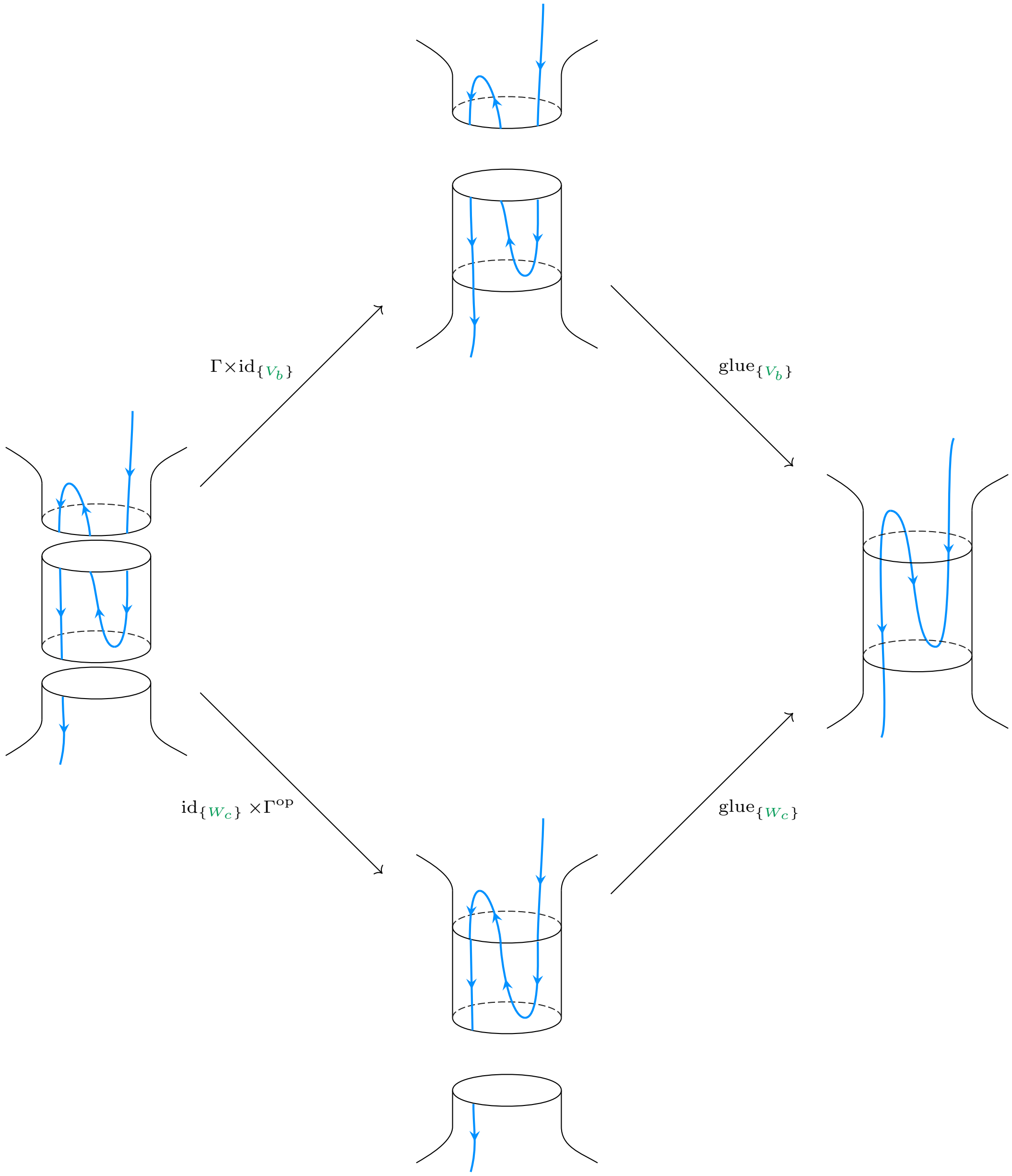
$$\xi, \omega \in \bigoplus_{\{U_d\}} \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{U_d\}^\vee, \{U_d\})$$

and any sensible notion of gluing should be able to detect that $\mathrm{glue}(\xi) \sim \mathrm{glue}(\omega)$, and the way one verifies this is in fact the case is by going through the statement in the last theorem. More generally, what is stated is that for any string-net state $\psi \in \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{V_b\}^\vee, \{W_c\})$ such that there exists a non-zero morphism $\Gamma \in \mathrm{Hom}_{\mathrm{Tube}_{\mathcal{C}}(S)}(\{V_b\}, \{W_c\})$, we can then identify:

$$\mathrm{glue}_{\{W_c\}}((\Gamma \times \mathrm{id}_{\{V_b\}}) \cdot \psi) = \mathrm{glue}_{\{W_c\}}((\mathrm{id}_{\{W_c\}^\vee} \times \Gamma^{\mathrm{op}}) \cdot \psi)$$

.To see how this applies to the example above let us try to find Γ and ψ such that:

$$(\Gamma \times \mathrm{id}_{\{V_b\}}) \cdot \psi = \xi$$
$$(\mathrm{id}_{\{W_c\}} \times \Gamma^{\mathrm{op}}) \cdot \psi = \omega$$



As a distinct way of phrasing the last theorem, we have the following corollary:

Corollary 2.16. *Let Σ and S be as before. Then we define the following subspace of $\bigoplus_{\{V_b\}} \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\big(\Sigma, \{V_b\}^{\vee}, \{V_b\}\big)$ which we will denote as the **sewing subspace** of Σ relative to $S \subset \partial\Sigma$:*

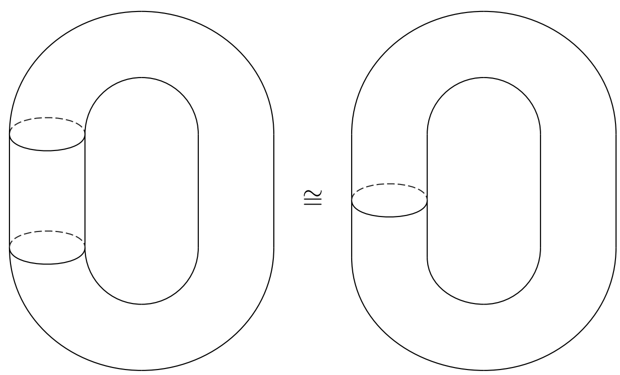
$$K := \ker(\mathrm{glue}) = \mathrm{Span}_{\mathbb{C}}\{\, \Gamma \cdot \psi - \psi \cdot \Gamma \}_{\Gamma, \psi} \tag{2.27}$$

where Γ ranges among $\mathrm{Hom}_{\mathrm{Tube}_{\mathcal{C}\,\mathcal{C}}(S)}\big(\{V_b\}, \{W_c\}\big)$ and ψ ranges over $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\big(\Sigma, \{V_b\}, \{W_c\}\big)$. Then the string-net *space of $\hat{\Sigma}_S$ is given as the quotient of $\bigoplus_{\{V_b\}} \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}\big(\Sigma, \{V_b\}^\vee, \{V_b\}\big)$ by the sewing subspace of Σ relative to S:*

$$\mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\hat{\Sigma}_S) \cong \left(\bigoplus_{\{V_b\}} \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma, \{V_b\}^\vee, \{V_b\}) \right) \Big/ K. \tag{2.28}$$

Pasting together this last string of results, it should be noted that we have achieved part of our goal in characterizing the space of string-nets. Namely, we have proved, using the machinery afforded to us by the gluing map and the action of the tube category, how to reconstruct the space of string-nets of a given surface by cutting it into simpler pieces and, with the knowledge of how they are defined on these simpler pieces, combine them with information on how to glue them, via sewing subspaces to recapture the global space. Not unlike how the usual resident of 221 Baker Street, London, "stitches" together collections of small evidence in his attempts at uncovering the identity of the city's most sought-after criminals.

It is not strange, in hindsight, that the tube category featured so prominently in our discussion about gluing manifolds along circles. After all, there is no distinction, in topological settings, between gluing two components of the boundary of a manifold or gluing a cylinder on one of the boundary components to be glued and then glue the end of that cylinder to the remaining boundary component. The figure below makes an attempt, by leaving the trace of where the gluing took place, to make visual the argument presented above.



The interesting notion, and one that unfortunately this chapter will be too small to fully grapple with, is the notion of introducing generalizations of the tube category for every manifold, not just circles, in an attempt to understand gluing rules of higher codimensions and in TQFTs of even higher dimension.

To conclude this section we prove a more algebraically flavoured version of the gluing *schemas* presented above. It characterizes, in the most computationally manageable form, the gluing operation we have been describing.

Theorem 2.17. *Let Σ and S be as above. Then we have that, for all $\{V_b\} \in \mathrm{Tube}_{\mathcal{C}}(S)$, there exists a linear map:*

$$\mathrm{glue}_{\{V_b\}} \colon \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma; \{V_b\}^\vee, \{V_b\}) \longrightarrow \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\hat{\Sigma}_S) \tag{2.29}$$

such that, for every morphism $\Gamma\colon \{V_b\} \longrightarrow \{W_c\}$ the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma;\{V_b\}^\vee,\{V_b\}) \arrow[dr, "\mathrm{glue}_{\{V_b\}}"] & \\
\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma;\{V_b\}^\vee,\{W_c\}) \arrow[ur, "\mathrm{id}_{\{V_b\}}\times\Gamma"] \arrow[dr, "\Gamma^{\mathrm{op}}\times\mathrm{id}_{\{W_c\}^\vee}"'] & & \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S) \\
& \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma;\{W_c\}^\vee,\{W_c\}) \arrow[ur, "\mathrm{glue}_{\{W_c\}}"'] &
\end{tikzcd} \tag{2.30}$$

Moreover, we have that:

$$\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S) = \int^{\{V_b\}\in\mathrm{Tube}_{\mathcal{C}\,\mathcal{C}}(S)} \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma;\{V_b\}^\vee,\{V_b\}) \tag{2.31}$$

where the gluing vector space $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S)$ corresponds to the coend of the functor:

$$\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma;-,-)\colon\ \mathrm{Tube}_{\mathcal{C}}(S)\times\mathrm{Tube}_{\mathcal{C}}(S)^{\mathrm{op}} \longrightarrow \mathrm{Vect} \tag{2.32}$$

Therefore, let (Q,glue') be a pair, where Q is a vector space and glue' is a map:

$$\mathrm{glue}'\colon \bigoplus_{\{V_b\}} \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma;\{V_b\}^\vee,\{V_b\}) \longrightarrow Q$$

for which the commutativity condition of diagram (16) is satisfied. Then there exists a unique isomorphism:

$$\zeta\colon\ \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S) \xrightarrow{\ \sim\ } Q \tag{2.33}$$

such that

$$\mathrm{glue}'_{\{V_b\}} = \zeta\circ\mathrm{glue}_{\{V_b\}} \qquad \textit{for all } \{V_b\}\in\mathrm{Tube}_{\mathcal{C}}(S) \tag{2.34}$$

In other words the following diagram commutes:

$$\begin{tikzcd}
& \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma;\{V_b\}^\vee,\{V_b\}) \arrow[dr, "\mathrm{glue}_{\{V_b\}}"] \arrow[drr, bend left, "\mathrm{glue}'_{\{V_b\}}"] & & \\
\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma;\{V_b\}^\vee,\{W_c\}) \arrow[ur, "\mathrm{id}_{\{V_b\}}\times\Gamma"] \arrow[dr, "\Gamma^{\mathrm{op}}\times\mathrm{id}_{\{W_c\}^\vee}"'] & & \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S) \arrow[r, dashed, "\zeta"] & Q \\
& \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma;\{W_c\}^\vee,\{W_c\}) \arrow[ur, "\mathrm{glue}_{\{W_c\}}"'] \arrow[urr, bend right, "\mathrm{glue}'_{\{W_c\}}"'] & &
\end{tikzcd} \tag{2.35}$$

Proof. Let $\Gamma \in \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\hat{\Sigma}_S)$, then we can, without any loss of generality, make the assumption that Γ, as an equivalence class, contains a representative that is transverse to the gluing submanifold S. As such, we assume that:

$$\{V_b\} = \Gamma|_S$$

which immediately implies the following equation:

$$\Gamma = \mathrm{glue}_{\{V_b\}}(\Gamma_{\mathrm{cut}}) \tag{2.36}$$

for some $\Gamma_{\mathrm{cut}} \in \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma; \{V_b\}, \{V_b\}^\vee)$. Considering the collection of equations in (20), which translate into the commutativity of the two triangles on the right of diagram (21), we must have the following definition for ζ:

$$\zeta(\mathrm{glue}_{\{V_b\}}) = \mathrm{glue}'_{\{V_b\}} \tag{2.37}$$

which means that if ζ is well-defined, then it is in fact unique, proving the first statement of the theorem. We are now left with proving that ζ is in fact a well-defined map, in other words: given a pair of string-nets $\Gamma, \theta \in \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\hat{\Sigma}_S)$ we should have that:

$$\Gamma = \theta \implies \zeta(\Gamma) = \zeta(\theta)$$

where, by consequence of our definition of ζ, the last statement is equivalent to:

$$\Gamma = \theta \implies \mathrm{glue}'_{\{V_b\}}(\Gamma_{\mathrm{cut}}) = \mathrm{glue}'_{\{W_c\}}(\theta_{\mathrm{cut}})$$

where

$$\{V_b\} = \Gamma|_S \qquad \text{and} \qquad \{W_c\} = \theta|_S$$

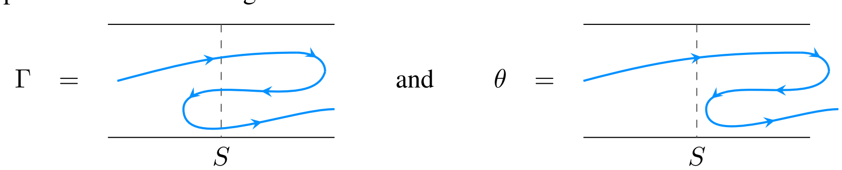
To do so, we will first strategically categorize the local relations via which Γ can be obtained from θ:

1. Isotopies in $\hat{\Sigma}_S$, which can be split into:
   (a) Isotopies enforced by disks $B \subset \hat{\Sigma}_S$, where $B \cap S = \varnothing$;
   (b) Collar shift isotopies along S.
2. Local relations with respect to disks $B \subset \hat{\Sigma}_S$ such that $B \cap S = \varnothing$;
3. Local relations with respect to disks $B \subset \hat{\Sigma}_S$ such that $B \cap S \neq \varnothing$.

For the 1(a) case, the proof is straightforward since if Γ and θ are related via an isotopy disjoint from S, then

$$\Gamma_{\mathrm{cut}} = \theta_{\mathrm{cut}}$$

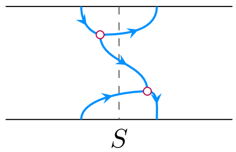
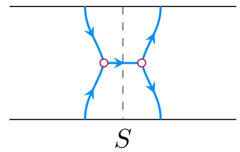
which immediately ensures that $\mathrm{glue}'_{\{V_b\}}(\Gamma_{\mathrm{cut}}) = \mathrm{glue}'_{\{W_c\}}(\theta_{\mathrm{cut}})$. For the 1(b) case, the collar shift isotopy, for which we provide an exemplification with the diagrams below:



then, by the hypothesis of the commutativity of the diagram (21), in particular its outer quadrilateral, we have that:

$$\mathrm{glue}'_{\{V_b\}}(\Gamma_{\mathrm{cut}}) = \mathrm{glue}'_{\{W_c\}}(\theta_{\mathrm{cut}}) \tag{2.38}$$

For case 2, the proof is as for 1(a). So we are left with checking the last case. As a means of providing visual aid to the proof, we appeal to following pair of examples for string-nets θ and Γ differing by a local relation on a disk B that intersects the gluing region:

$\Gamma \quad =$  and $\quad \theta \quad =$ 

In this case, we can pick an isotopy $\mathrm{Iso}\colon \hat{\Sigma}_S \to \hat{\Sigma}_S$ of $\hat{\Sigma}_S$, such that:

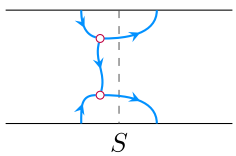
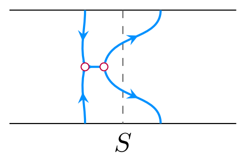
$$\mathrm{Iso}(B) \cap S \neq \varnothing$$

the disk on which we mod out the local relation is now disjoint of the gluing "seam" S. As a consequence we get the following important equations:

$$\mathrm{Iso}_*(\Gamma) = \Gamma \tag{2.39}$$

$$\mathrm{Iso}_*(\theta) = \theta \tag{2.40}$$

where the Iso_* is the induced pushforward associated to the diffeoemorphism Iso. The diagrams below, showcase an example where we track the action of such an isotopy on the diagrams Γ and θ above:

$\Gamma \quad =$  and $\quad \theta \quad =$ 

We define:

$$\{V'_{b'}\} = \mathrm{Iso}_*(\Gamma)|_S \tag{2.41}$$

$$\{W'_{c'}\} = \mathrm{Iso}_*(\theta)|_S \tag{2.42}$$

Notice that, as a consequence of the action of the isotopy, we have that

$$\mathrm{Iso}_*(\Gamma) = \mathrm{Iso}_*(\theta) \tag{2.43}$$

and they are related via an isotopy applied on a disk disjoint to S, meaning that they are now an example of case 1(a) and hence implying that:

$$\mathrm{glue}'_{\{V'_{b'}\}}(\mathrm{Iso}_*(\Gamma)_{\mathrm{cut}}) = \mathrm{glue}'_{\{W'_{c'}\}}(\mathrm{Iso}_*(\theta)_{\mathrm{cut}}) \tag{2.44}$$

We can combine the last four equations, **??**, (26), (27) and (28), in order to conclude that:

$$\mathrm{glue}'_{\{V_b\}}(\Gamma_{\mathrm{cut}}) = \mathrm{glue}'_{\{V'_{b'}\}}(\mathrm{Iso}_*(\Gamma)_{\mathrm{cut}}) = \mathrm{glue}'_{\{W'_{c'}\}}(\mathrm{Iso}_*(\theta)_{\mathrm{cut}}) = \mathrm{glue}'_{\{W_c\}}(\theta_{\mathrm{cut}}) \tag{2.45}$$

where the first and last equality come from the fact that $\mathrm{Iso}_*(\Gamma)$ and $\mathrm{Iso}_*(\theta)$ are related to Γ and θ by an arbitrary isotopy in $\hat{\Sigma}_S$, respectively, and by evoking the proof for cases 1(a)-(b) and 2 finishes the proof. $\square$

While taking in this highly algebraic description of the locality of the String-Net vector space, one should expect that given a surface Σ with two pairs $(S, \overline{S})$ and $(Q, \overline{Q})$ of identified boundary components, and since

$$\hat{\Sigma}_{S,Q} \cong \hat{\Sigma}_{Q,S},$$

where $\hat{\Sigma}_{S,Q}$ is the surface obtained from gluing first the $(S, \overline{S})$ component and then the $(Q, \overline{Q})$ component, while $\hat{\Sigma}_{Q,S}$ is defined in the reverse order, by first attaching the Q and the S components respectively, then:

$$\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_{S,Q}) \cong \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_{Q,S}). \tag{2.46}$$

This is ensured by a the following Fubini-type theorem for coends.

Theorem 2.18. *Given a functor*

$$F\colon \mathcal{C} \times \mathcal{C}^{\mathrm{op}} \times \mathcal{D} \times \mathcal{D}^{\mathrm{op}} \longrightarrow \mathrm{Vect}$$

for which the coends

$$\int^{c\in\mathcal{C}} F(c, c, d, d') \quad \textit{as well as the coends} \quad \int^{d\in\mathcal{D}} F(c, c', d, d)$$

exist for all $c, c' \in \mathcal{C}$ and $d, d' \in \mathcal{D}$, respectively. There exist unique isomorphisms

$$\int^{c\in\mathcal{C}} \left(\int^{d\in\mathcal{D}} F(c, c, d, d) \right) \cong \int^{c\times d\in\mathcal{C}\times\mathcal{D}} F(c, c, d, d) \cong \int^{d\in\mathcal{D}} \left(\int^{c\in\mathcal{C}} F(c, c, d, d) \right)$$

Proof. A proof for this result can be found in Mac Lane [19, $Ch.$ IX] for a slightly more general setting. □

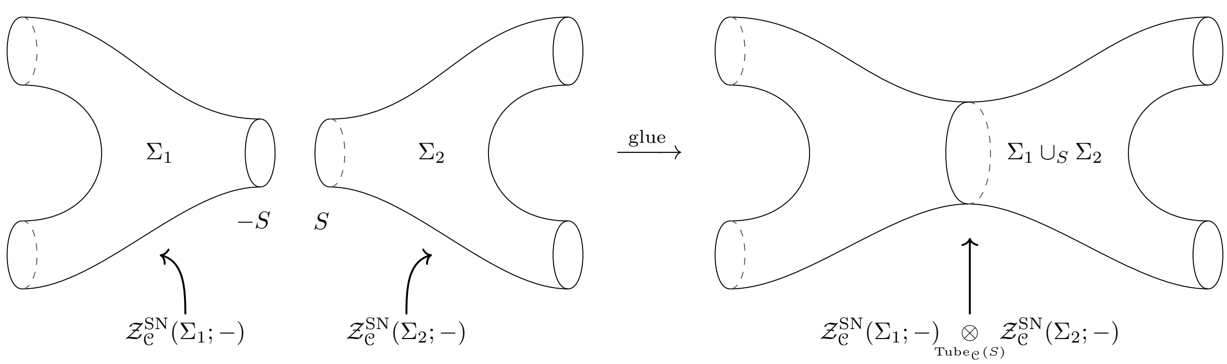
In the context of String-Nets, this means the coend construction captures the fact that the topology of the surface is not altered by choosing a different gluing order, so-to-speak, and serves as a consistency check. The next corollary provides a description of the gluing scheme via coends in the case that Σ is disjoint and the glued boundaries belong to distinct components of Σ.

Corollary 2.19. *Let Σ_1 and Σ_2 be a pair of 2-dimensional cobordism such that:*

$$\partial\Sigma_1 = R \sqcup \overline{S}$$
$$\partial\Sigma_2 = S \sqcup -Q$$

then we have that:

$$\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma_1 \cup_S \Sigma_2) \cong \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma_1; -) \underset{\mathrm{Tube}_{\mathcal{C}}(S)}{\otimes} \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma_2; -) \tag{2.47}$$



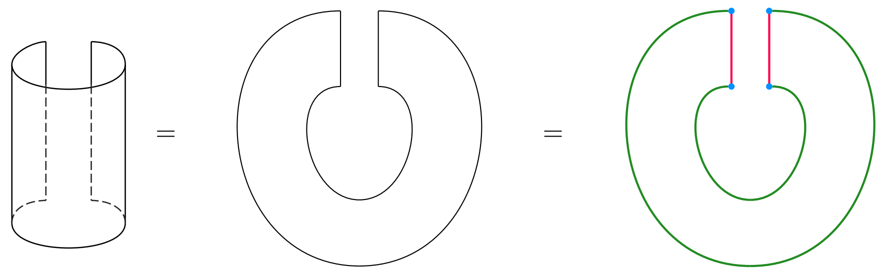
2.4 Tube Categories for the Interval

Note, that this section is mainly a sketch and no attempt at rigour is really made.

With this subsection, we aim at providing a quick summary of how the above results generalize to a slightly more general class of surfaces. We noted in the introduction how the String-Net TQFTs should be fully-extended TQFTs [24][20] since they are constructed from spherical fusion categories, which are conjectured to be the $\mathrm{SO}(3)$ homotopy fixed-points in the 3-category of $\mathbb{C}$ linear monoidal categories.

Although attempting to fully-extend this construction would be too long to include here, since it would inevitably lead us to the need for understanding how the 2-category associated to a point can be leveraged to reconstruct the 1-categories associated to 1-manifolds and introduce potentially more complicated gluing rules, it is a trademark of any fully-extended TQFT that one must know how to label every 1-manifold with a 1-category. In the case of String-Nets, it turns out that doing this is rather processional since the 2-category with which we start has only one object implying then that the point is always labelled with the unique object $\mathcal{C}$.

Understanding this, we can extend the construction String-Nets from surfaces with boundary to surfaces with corners. Consider the following surface with corners Σ and let us use it as a running example:

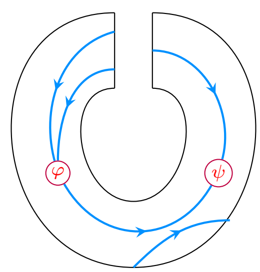


$$. \tag{2.48}$$

We decompose its boundary as follows

$$\partial\Sigma = \overline{S} \cup S \cup R \tag{2.49}$$

where $\overline{S}$ and S are represented by the red submanifold and the R is the green submanifold. Moreover we have that $\partial R = P \sqcup \overline{P}$ where P and $\overline{P}$ are the 0-dimensional submanifolds represented by the blue dots. As an example of a String-Net in Σ we have the figure given below:



(2.50)

The only condition we need to extend the String-Net vector space for surfaces with corners is that corners must remain unlabeled (or trivially labelled by $\mathcal{C}$, the only object in a 2-category with one object). All TQFT properties are satisfied, but the boundary conditions need to be extended to include not just oriented labelled circles, but also oriented labelled intervals. We recycle the notation for circular boundary conditions since it will always be clear from context which one is being used. Having said this, we are immediately prompted to generalize our construction of the tube category to include $\mathrm{Tube}_{\mathcal{C}}(I)$ where I is an interval.

Definition 2.20. Let I be an oriented interval. We define $\mathrm{Tube}_{\mathcal{C}}(I)$ to be category whose objects are intervals labelled with objects from $\mathcal{C}$, except for the end-points. A pair of examples are given by :

$\{V_b\}_{b\in B} =$ 

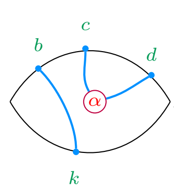
, $\qquad \{W_c\}_{c\in C} =$ 

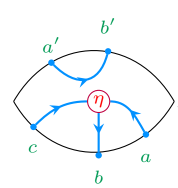
. (2.51)

The morphisms between two boundary conditions $\{V_b\}_{b\in B}$ and $\{W_c\}$ are defined to be the vector space of String-Nets in the pinched interval cobordism $I \times_{\mathrm{p}} I$ i.e. a bigon:

$$\mathrm{Hom}_{\mathrm{Tube}_{\mathcal{C}}(I)}(\{V_b\}, \{W_c\}) := \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(I \times_{\mathrm{p}} I; \{W_c\}, \{W_c\}^\vee) \tag{2.52}$$

To exemplify, we have:

$\Gamma_1 =$ 

, $\qquad \Gamma_2 =$ 

, (2.53)

and composition is given by gluing along I:

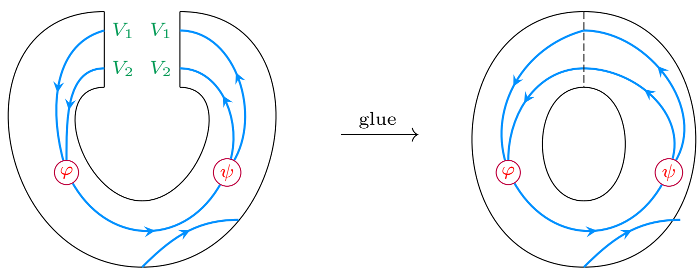
$$\Gamma_1 \circ \Gamma_2 \ = \tag{2.54}$$

Since we do not pretend to mimic the entire discussion of the previous subsection for the case of the tube category of the interval, we list some of the key points of convergence and disticntion:

1. $\mathrm{Tube}_{\mathcal{C}}(\overline{I}) = \mathrm{Tube}_{\mathcal{C}}(I)^{\mathrm{op}}$;
2. Given a pair of oriented intervals I and J we have that $\mathrm{Tube}_{\mathcal{C}}(I \sqcup J) = \mathrm{Tube}_{\mathcal{C}}(I) \times \mathrm{Tube}_{\mathcal{C}}(J)$;
3. The tube category of an interval I acts on the space of string-nets of a surface with corners where one of the boundary components can be identified with I, and assuming that boundary conditions for all other boundary components are fixed;
4. Let Σ be an oriented manifold with corners and a pair of identifiable boundary components $I, \overline{I}$ such that $\partial I = P$, where P is the disjoint union of a pair of 0-manifolds. In this case, there is a gluing map, in all ways similar to the one defined for the tube category of the circle:

$$\mathrm{glue}\colon \bigoplus_{\{V_b\}} \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; \{V_b\}, \{V_b\}^{\vee}) \longrightarrow \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma), \tag{2.55}$$

which is exemplified in the following operation:



(2.56)

5. Given a surface Σ with corners, such that

$$\partial\Sigma = \overline{S} \cup S \cup R, \tag{2.57}$$

and suppose that $\partial S = P$ and that $\partial R = P \sqcup \overline{P}$. If we glue Σ along S to get $\hat{\Sigma}_S$, then we get that $\partial\hat{\Sigma}_S$ is just $\hat{R}_P$, corresponding to R with $\overline{P}$ and P glued. Diagram 2.56 serves as the source example for this abstract decomposition. Now, fixing a boundary condition $\{V_b\}$ for $\hat{R}_P$ i.e. a pair of circles corresponding to the gluing of a boundary condition for R i.e. a pair of intervals $\{V_b\}_{\mathrm{cut}}$. Then we have that $\mathrm{Tube}_{\mathcal{C}}(S)^{\mathrm{op}} \times \mathrm{Tube}_{\mathcal{C}}(S)$ acts on the space $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; -, -, \{V_b\})_{\mathrm{cut}}$ of String-Nets in Σ with a fixed boundary condition for R. Then, mimicking the result for tube categories of the circle we have that $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\hat{\Sigma}_S, \{V_b\})$ is the coend of the action of $\mathrm{Tube}_{\mathcal{C}}(S)^{\mathrm{op}} \times \mathrm{Tube}_{\mathcal{C}}(S)$ on $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma; -, -, \{V_b\})_{\mathrm{cut}}$.

Although we do not prove this last point, we want to note that a proof for it follows from our proof of 2.17 with little change since our reasoning applies almost *mutatis mutandis* to the case of above.

2.5 String-Nets on Punctured Surfaces

In this subsection we closely follow the work of Kirillov Jr. [13], we add missing details and adapt the results there to our conventions and terminology. The goal is to characterize the space of String-Nets as the image of a projector defined on the space of String-Nets of a surface with a finite set of points removed i.e. a finite set of punctures. Its importance for the greater goal of the thesis, besides adding robustness to our understanding of String-Nets, is that the space of string-nets on a punctured surface can be canonically identified with the state-space of Turaev-Viro and with the image of an important set of projectors, called vertex projectors, in Levin-Wen Hamiltonians.

We begin our discussion by characterizing the space of String-Nets in a once-punctured surface through the following lemma:

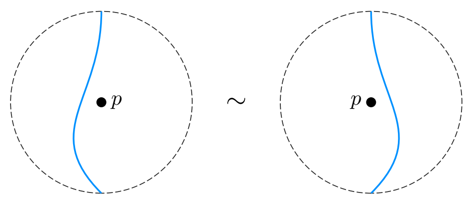
Lemma 2.21. *Let Σ be a closed oriented surface and let $p \in \Sigma$ be an arbitrary point. Then there is an embedding:*

$$\iota\colon \mathcal{H}_{\mathcal{C}}(\Sigma - p) \hookrightarrow \mathcal{H}_{\mathcal{C}}(\Sigma)$$

which descends into an isomorphism:

$$\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma - p)/\sim\ \cong\ \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma) \tag{2.58}$$

where the relation $\sim$ in $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma - p)$ is given by including the following local relation:



Proof. The proof results from the observation that all String-Nets on a closed surface with a puncture can be recast as String-Nets in the same surface with the puncture included, if we lose the local relations containing isotopies in which any strand intersects the puncture point. In other words, by including the possibility of isotoping a strand past the puncture in $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma - p)$, we immediately recover the entire space $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma)$ of String-Nets in Σ. The result is simple to state and prove, but it is an insightful observation. □

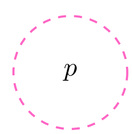
We now make use of the lemma to establish an important result in understanding the connection between the space of string-nets on a punctured surface versus its space of string-nets on the same unpunctured surface. To do so, we with a defintion and theorem;

Theorem 2.22. *Let Σ be a closed oriented surface and let p be an arbitrary point of Σ, we define the Cloaking operator:*

$$\mathrm{Cloak}_p\colon \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma - p) \longrightarrow \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma - p) \tag{2.59}$$

to be the operator that adds to every labeled graph in $\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma - p)$ a cloaking element around the puncture p and multiplies it by a factor of $\dfrac{1}{\mathcal{D}^2}$. It is graphically represented as:

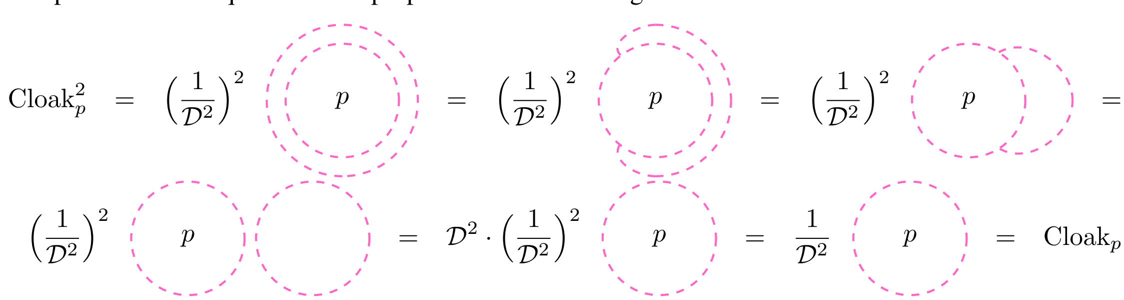
$$\mathrm{Cloak}_p \quad = \quad \frac{1}{\mathcal{D}^2}$$



Moreover, the operator Cloak_p is a projector, meaning:

$$\mathrm{Cloak}_p^2 = \mathrm{Cloak}_p \tag{2.60}$$

Proof. The proof is a consequence of the properties of the cloaking element:



where we take some liberties in notation given that all the vertices in the second and third diagrams should be labeled by the basis and dual basis elements of their respective morphisms spaces. Other than that, the proof comes from local relations presented in the earlier part of this section. $\square$

Now let us consider these cloaking operators in the context of additional punctures. Then we have the following result:

Lemma 2.23. *Let Σ be a closed oriented surface and let $P := \{p_1, \dots, p_k\}$ be a finite set for some $k \in \mathbb{N}$, of distinct points $p_i \in \Sigma$. Then the set of operators $\{\mathrm{Cloak}_{p_i}\}_{p_i \in P}$ all commute with each other and as consequence, the operator:*

$$\mathrm{Cloak}_P := \prod_{p \in P} \mathrm{Cloak}_p \colon \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma - P) \longrightarrow \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma - P) \tag{2.61}$$

is also a projector.

Proof. The operators commute with each other as a consequence of the fact that the cloaking elements that are added to each puncture, regardless of the order in which they are added to any labeled graph, do not "interfere" with each other since they are placed around the p_i's guaranteeing that only the points p_i lie inside it and hence making their placement's order of irrelevance to the final state. The last statement comes from the fact the product of set of commuting projectors is still a projector, to see why we can, due to the commutativity, rearrange the factors in Cloak_P^2, so that $\mathrm{Cloak}_P^2 = \mathrm{Cloak}_{p_1}^2 \cdots \mathrm{Cloak}_{p_k}^2 = \mathrm{Cloak}_P$, finishing the proof. $\square$

Which leads us to the important final result:

Theorem 2.24. *Let Σ be a closed oriented surface and let $P := \{p_1, \dots, p_k\}$ be a finite set for some $k \in \mathbb{N}$, of distinct points $p_i \in \Sigma$. Then we have the following isomorphism:*

$$\mathrm{Im}(\mathrm{Cloak}_P) = \{\Gamma \in \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma - P) \colon \mathrm{Cloak}_P \cdot \Gamma = \Gamma\} \cong \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma) \tag{2.62}$$

Proof. The first equality:

$$\mathrm{Im}(\mathrm{Cloak}_P) = \{\Gamma \in \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma - P) \colon \mathrm{Cloak}_P \cdot \Gamma = \Gamma\}$$

is a direct consequence of the previous theorem. Since Cloak_P is a projector we have that if $\Gamma' = \mathrm{Cloak}_P \cdot \Gamma$ i.e. $\Gamma' \in \mathrm{Im}(\mathrm{Cloak}_P)$, then:

$$\mathrm{Cloak}_P \cdot \Gamma' = \mathrm{Cloak}_P^2 \cdot \Gamma = \mathrm{Cloak}_P \cdot \Gamma = \Gamma'$$

where the first equality is just the definiton of Γ', the second comes from $\mathrm{Cloak}_P^2 = \mathrm{Cloak}_P$ and the last one is, again, just the definition of Γ'. This settles the first equation, for the second equation let us recall that:

$$\mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma - P)/\sim \;\cong\; \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma)$$

and as such there is a surjective, quotient map:

$$\tilde{\iota} \colon \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma - P) \twoheadrightarrow \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma)$$

which prompts the question whether $\mathrm{Cloak}_P \colon \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma - P) \longrightarrow \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma - P)$ is well-defined on the quotient or in other words whether the map:

$$\begin{aligned} \widetilde{\mathrm{Cloak}}_P \colon \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma) &\longrightarrow \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma) \\ [\Gamma] &\longmapsto [\mathrm{Cloak}_P \cdot \Gamma] \end{aligned}$$

where $[\Gamma]$ corresponds to the equivalence class associated to $\Gamma \in \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma - P)$ under the isomorphism:

$$\left(\mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma - P)/\sim\right) \;\cong\; \mathcal{Z}_{\mathcal{C}}^{\mathrm{SN}}(\Sigma)$$

is a well-defined map, and indeed the answer is yes, since the map does not depend on which representative we pick. To see why this is true notice that if $[\Gamma_1] = [\Gamma_2]$ then it does not affect change equivalence classes to first cloak the set of points in P since $[\Gamma_1]$ and $[\Gamma_2]$ are already indistinguishable under istopies of Σ the move strands across any point $p_i \in P$. We can synthesize all of this into the statement that:

$$\begin{tikzcd}
\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma - P) \arrow[r, "\mathrm{Cloak}_P"] \arrow[d, "\tilde{\iota}"'] & \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma - P) \arrow[d, "\tilde{\iota}"] \\
\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma) \arrow[r, "\widetilde{\mathrm{Cloak}_P}"'] & \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma)
\end{tikzcd} \tag{2.63}$$

is a commutative diagram. Moreover we have that:

$$\mathrm{Im}(\mathrm{Cloak}_{\mathrm{P}}) \cong \tilde{\iota}\big(\,\mathrm{Im}(\mathrm{Cloak}_P)\big)$$

stemming from the fact that $\tilde{\iota}|_{\mathrm{Im}(\mathrm{Cloak}_p)} \colon \mathrm{Im}(\mathrm{Cloak}_p) \longrightarrow \tilde{\iota}\big(\,\mathrm{Im}(\mathrm{Cloak}_P)\big)$ i.e. the restriction of the quotient to the image of Cloak_P, is injective and it is surjective hence establishing witnessing an honest isomorphism. The surjectivity is a consequence of the defintion while the injectivity still needs checking. Let us assume that $\Gamma_1, \Gamma_2 \in \mathrm{Im}(\mathrm{Cloak}_P)$ and that:

$$\begin{aligned}
\tilde{\iota}(\Gamma_1) = \tilde{\iota}(\Gamma_2) && \Leftrightarrow \\
\tilde{\iota}(\mathrm{Cloak}_P \cdot \Gamma_1) = \tilde{\iota}(\mathrm{Cloak}_P \cdot \Gamma_2) && \Leftrightarrow \\
\tilde{\iota}(\mathrm{Cloak}_P \cdot \Gamma_1 - \mathrm{Cloak}_P \cdot \Gamma_2) = 0 && \Leftrightarrow \\
\mathrm{Cloak}_P \cdot (\Gamma_1 - \Gamma_2) \in \ker(\tilde{\iota}) \Rightarrow \mathrm{Cloak}_P^2 \cdot (\Gamma_1 - \Gamma_2) = 0 && \Leftrightarrow \\
\mathrm{Cloak}_P \cdot (\Gamma_1 - \Gamma_2) = 0 && \Leftrightarrow \\
\Gamma_1 - \Gamma_2 = 0
\end{aligned}$$

where from the first to the second line we used our previous characterization of elements in $\mathrm{Im}(\mathrm{Cloak}_P)$, the implication on the fourth line comes from our previous discussion in Lemma 6.2, it can be summarized by $\Gamma \in \ker(\tilde{\iota}) \Rightarrow \mathrm{Cloak}_P \cdot \Gamma = 0$. The last two lines come from Cloak_P being a projector and $\Gamma_1, \Gamma_2 \in \mathrm{Im}(\mathrm{Cloak}_P)$, respectively. To put an end to the proof all that we are left to check is that:

$$\tilde{\iota}\big(\,\mathrm{Im}(\mathrm{Cloak}_P)\big) = \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma)$$

To understand why it is true we make use of the commutatvity of diagram (31) to say that:

$$\tilde{\iota}\big(\,\mathrm{Im}(\mathrm{Cloak}_P)\big) = \mathrm{Im}(\widetilde{\mathrm{Cloak}_P}) \tag{2.64}$$

and since $\widetilde{\mathrm{Cloak}_P} \cdot [\Gamma] = [\mathrm{Cloak}_P \cdot \Gamma] = [\Gamma]$, then we have that $\widetilde{\mathrm{Cloak}_P} = \mathrm{id}_{\mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma)}$ and as such:

$$\mathrm{Im}(\widetilde{\mathrm{Cloak}_P})) = \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma) \tag{2.65}$$

finally concluding our proof that:

$$\mathrm{Im}(\mathrm{Cloak}_P) \cong \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\Sigma)$$

$\square$

3 Modules of the Tube Category

In this section we focus exclusively on the goal of proving that the category of modules of the tube category $\mathrm{Tube}_{\mathcal{C}}(S^1)$ is contravariantly equivalent to the Drinfel'd center of $\mathcal{C}$. This result is typically stated as a folklore result in the "halls" of TQFT and there are a multitude of of paths leading towards its proof, each with its own merits. In Walker's lecture notes [24] (which the author was highly inspired by), a sketch of proof for this result is given by taking the fully-extended data of the String-Net TQFT and comparing the Drinfel'd center of $\mathcal{C}$ with the categorified end of a 2-module for a bicategory with one object. In Kirillov's equally ground-breaking work on Extended Turaev-Viro [11], a proof for this result is proposed although with a different flavour. Kirillov proves that $\mathcal{Z}(\mathcal{C}) \simeq \mathrm{Kar}(\mathrm{Tube}_{\mathcal{C}}(S^1))$, where $\mathrm{Kar}(\mathcal{C})$ is the Karoubi completion (also called Idempotent completion) of $\mathcal{C}$. In particular, his proof omits the balanced structure associated to $\mathrm{Kar}(\mathcal{C})$ and hence fails to establish that the equivalence is braided monoidal. More recently, through the work of Hardiman [8], Hoek [9] and Goosen [7] a fully graphical approach to this proof has emerged that embeds it in the context of fully-extended TQFTs while simultaneously preserving the interpretation of provided by the Karoubi completion suggested by Kirillov. Our version of this proof is really an amalgam of all the other proofs.

Before starting the proof, we collect some basic results we will need concerning the category of modules of a semisimple category by introduction the Yoneda lemma and embedding and we recall how to construct the Drinfel'd center of a monoidal category.

3.1 Karoubi completion and the Yoneda lemma

In this we follow the discussion in Hardiman's article [8], and we introduce the notion of Karoubi completion, the quintessential pair of the Yoneda lemma and the Yoneda embedding, also we show that for a linear category $\mathcal{C}$ its category of modules is equivalent to its Karoubi completions $\mathrm{Kar}(\mathcal{C})$ and we show that if $\mathcal{C}$ is semisimple then $\mathrm{Kar}(\mathcal{C}) \simeq \mathcal{C}$.

We begin with a sketch of proof for the Yoneda lemma:

Theorem 3.1. *(Yoneda Lemma) Let $\mathcal{C}$ be a $\mathbb{C}$-linear category and $F\colon \mathcal{C} \longrightarrow \mathrm{Vect}_{\mathbb{C}}$ be a linear functor in $\mathrm{Mod}_{\mathcal{C}}$. Then the following vector spaces are isomporphic:*

$$\mathrm{Nat}\big(\mathrm{Hom}_{\mathcal{C}}(V,-), F\big) \cong F(V) \tag{3.1}$$

Proof. We will not not give a formal proof for this result. Instead, we provide a small sketch. Given an arbitrary natural transformation

$$\eta\colon\ \mathrm{Hom}_{\mathcal{C}}(V,-) \Longrightarrow F,$$

and given an arbitrary morphism

$$\varphi\colon V \longrightarrow W,$$

in $\mathcal{C}$. The naturality of η is summed up by the commutativity of the following diagram:

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(V,V) \arrow[r, "\eta_V"] \arrow[d, "\varphi\circ-"'] & F(V) \arrow[d, "F(\varphi)"] \\ \mathrm{Hom}_{\mathcal{C}}(V,W) \arrow[r, "\eta_W"'] & F(W). \end{tikzcd}$$

Chasing the identity of V around the diagram we end up with the follwoing equation:

$$\eta_W(\varphi) = F(\varphi)(\eta_V(\mathrm{id}_V)). \tag{3.2}$$

Which guarantees that all components η_W of this natural transformation are determined by the value of $F(\varphi)$ at $\eta_V(\mathrm{id}_V)$ which takes values in $F(V)$. On the other hand, we see that vectors $v \in F(V)$ produce a family of linear maps

$$\eta_W(\varphi \circ \mathrm{id}_V) := F(\varphi)(v)$$

which satisfy the naturality condition. $\square$

Also, of great interest to us in the subsequent subsections is the following corollary, or specialization, of the Yoneda lemma, focusing in the particular case when $F = \mathrm{Hom}_{\mathcal{C}}(W, -)$ is also a Hom-functor.

Corollary 3.2. *(Yoneda embedding) There is a fully faithful functor*

$$\mathcal{Y}\colon \mathcal{C}^{\mathrm{op}} \longrightarrow \mathrm{Fun}(\mathcal{C}, \mathrm{Vect}), \tag{3.3}$$

defined by mapping

$$V \longmapsto \mathrm{Hom}_{\mathcal{C}}(V, -) \quad \textit{and} \quad (\varphi\colon V \to W) \longmapsto \eta$$

where $\eta \in \mathrm{Nat}\big(\mathrm{Hom}_{\mathcal{C}}(V, -), \mathrm{Hom}_{\mathcal{C}}(W, -)\big)$ *is the unique natural transformation represented by* $\mathrm{Hom}_{\mathcal{C}}(W, V)$*, under the Yoneda lemma.*

Proof. Considering the statement of the Yoneda lemma for $F = \mathrm{Hom}_{\mathcal{C}}(W, -)$, we get that:

$$\mathrm{Nat}\big(\mathrm{Hom}_{\mathcal{C}}(V, -), \mathrm{Hom}_{\mathcal{C}}(W, -)\big) \cong \mathrm{Hom}_{\mathcal{C}}(W, V) \tag{3.4}$$

Which is equivalent to the statement that $\mathcal{Y}$ is fully faithful. Moreover, unwrapping formula 3.2 with $F = \mathrm{Hom}_{\mathcal{C}}(W, -))$ we get:

$$\eta_W(\varphi) = \mathrm{Hom}_{\mathcal{C}}(V, \varphi)(\eta_V(\mathrm{id}_V)) = \varphi \circ \eta_V(\mathrm{id}_V). \tag{3.5}$$

$\square$

Now we introduce the concept of idempotent completion or Karoubi completion.

Definition 3.3. Let $\mathcal{C}$ be a category. We define the **Karoubi completion** of $\mathcal{C}$ to be the pair $(\mathrm{Kar}(\mathcal{C}), \Omega)$, where:

1. $\Omega\colon \mathcal{C} \longrightarrow \mathrm{Kar}(\mathcal{C})$ is a covariant fully faithful fucntor;
2. $\mathrm{Kar}(\mathcal{C})$ is idempotent complete (see Defintion **??**);
3. For every object V in $\mathcal{C}$ there exists and idempotent p in $\mathcal{C}$ such that V is an image object for $\Omega(p)$.

Proposition 3.4. *The subcategory* $\mathrm{Kar}(\mathcal{C})$ *of* $\mathcal{C}$ *coupled with the Yoneda embedding is an idempotent completion of* $\mathcal{C}$*.*

Proof. For a proof see Hardiman's article [8] or Lurie's book [17, Sec. 5.4]. $\square$

Proposition 3.5. *If* $\mathcal{C}$ *is a semisimple category, then:*

$$\mathcal{C} \simeq \mathrm{Fun}(\mathcal{C}, \mathrm{Vect}). \tag{3.6}$$

With the equivalence witnessed via the Yoneda embedding.

Proof. The Yoneda embedding is fully faithful as was proved in Corollary 3.2, hence we only have to prove that it is also essentially surjective. For all $V_j \in \mathrm{Irr}(\mathcal{C})$ we have that:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}\left(\bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} F(V_i) \cdot V_i, V_j\right) &\cong \bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathcal{C}}(F(V_i) \cdot V_i, V_j) \\
&\cong \bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}_{\mathrm{Vect}}(F(V_i), \mathrm{Hom}_{\mathcal{C}}(V_i, V_j)) \\
&\cong \mathrm{Hom}_{\mathrm{Vect}}(F(V_j), \mathrm{Hom}_{\mathcal{C}}(V_j, V_j)) \\
&\cong \mathrm{Hom}_{\mathrm{Vect}}(F(V_j), \mathbb{C}) \\
&\cong F(V_j)
\end{aligned}$$

and since F is completely determined by its image on the simple objects of $\mathcal{C}$ we establish that

$$F \cong \mathcal{Y}\left(\bigoplus_{i \in \mathrm{Irr}(\mathcal{C})} F(V_i) \cdot V_i \right). \tag{3.7}$$

□

Remark 3.6. We note that, $\mathrm{Fun}(\mathcal{C}, \mathrm{Vect})$ is also the category of modules of $\mathcal{C}$, and with the result in Proposition 3.5 we get that a semisimple category corresponds to its own idempotent completion, as would expected, and that it is equivalent to its representation category.

3.2 The Drinfel'd center

The Drinfel' center is one the most important bits in the puzzle of this section. It was constructed as a way of turning a monoidal category $\mathcal{C}$ into a braided monoidal category $\mathcal{Z}(\mathcal{C})$ [10]. In Müger's seminal paper [21], he proves that if $\mathcal{C}$ is not just a monoidal category but a spherical fusion category, then $\mathcal{Z}(\mathcal{C})$ is not just braided but it is also a modular tensor category[2], one which is, in particular, anomaly-free. The name of the construction is suggestive in the sense that the Drinfel'd center can be recast as a categorification of the center of a monoid.

Definition 3.7. (Drinfel'd Center) Let $(\mathcal{C}, \otimes, a, 1, \ell, r)$ be a monoidal category, we define the *Drinfel'd center* of $\mathcal{C}$ as the category $\mathcal{Z}(\mathcal{C})$ whose objects are given by pairs (X, c) where X is an object of $\mathcal{C}$ and

$$c_{X,Z} \colon Z \otimes X \to X \otimes Z \tag{3.8}$$

is a natural isomrphism for which the following diagram commutes:

$$\begin{tikzcd}
 & X \otimes (Y \otimes Z) \arrow[dl, "a^{-1}_{X,Y,Z}"'] \arrow[dr, "\mathrm{id}_X \otimes c_Y"] & \\
(X \otimes Y) \otimes Z \arrow[d, "c_{X \otimes Y}"'] & & X \otimes (Z \otimes Y) \arrow[d, "a^{-1}_{X,Z,Y}"] \\
Z \otimes (X \otimes Y) \arrow[dr, "a^{-1}_{Z,X,Y}"'] & & (X \otimes Z) \otimes Y \arrow[dl, "c_X \otimes \mathrm{id}_Y"] \\
 & (Z \otimes X) \otimes Y &
\end{tikzcd} \tag{3.9}$$

with morphisms $f \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}((Z, c), (Z', c'))$ given as the subset of $f \in \mathrm{Hom}_{\mathcal{C}}(Z, Z')$ satisfying the constraint:

$$(f \otimes \mathrm{id}_X) \circ_X = c'_X \circ (\mathrm{id}_X \otimes f) \tag{3.10}$$

for all $X \in \mathrm{Obj}(\mathcal{C})$.

On it, we can define a monoidal structure for $\mathcal{Z}(\mathcal{C})$:

Theorem 3.8. *Let $\mathcal{Z}(\mathcal{C})$ be the Drinfel'd center associated to a monoidal category $\mathcal{C}$. Then the following bifunctor $\otimes \colon \mathcal{Z}(\mathcal{C}) \times \mathcal{Z}(\mathcal{C}) \longrightarrow \mathcal{Z}(\mathcal{C})$:*

$$(X, c) \otimes (Y, c') := (X \otimes Y, \tilde{c}) \tag{3.11}$$

where

$$\tilde{c}_Z \colon Z \otimes (X \otimes Y) \longrightarrow (X \otimes Y) \otimes Z \tag{3.12}$$

defined by the commutativity of the following diagram:

$$\begin{tikzcd}
Z \otimes (X \otimes Y) \arrow[r, "a^{-1}_{Z,X,Y}"] \arrow[ddr, "\tilde{c}_Z"'] & (Z \otimes X) \otimes Y \arrow[r, "c_Z \otimes \mathrm{id}_Y"] & (X \otimes Z) \otimes Y \arrow[d, "a_{X,Z,Y}"] \\
 & & X \otimes (Z \otimes Y) \arrow[d, "\mathrm{id}_X \otimes c_Z"] \\
 & (X \otimes Y) \otimes Z & X \otimes (Y \otimes Z) \arrow[l, "a^{-1}_{X,Y,Z}"]
\end{tikzcd} \tag{3.13}$$

with the monoidal unit $(1, c)$ *defined as:*

$$\begin{tikzcd}
X \otimes 1 \arrow[r, "\ell_X"] \arrow[rr, bend right, "c"'] & X \arrow[r, "r_X^{-1}"] & 1 \otimes X
\end{tikzcd} \tag{3.14}$$

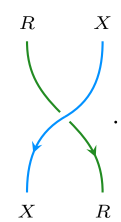
is a monoidal category. Additionally, we define the following braiding:

$$c_{(X,c),(Y,c')} := c'_X \tag{3.15}$$

ensuring that $\mathcal{Z}(\mathcal{C})$ *is not only a monoidal category but a braided monoidal category.*

Since we will consider Drinfel'd centers taken from spherical fusion category, the result of Müger that $\mathcal{Z}(\mathcal{C})$ is a modular tensor categpry, coupled with Reshetikhin and Turaev's work [22] imply that the Drinfel'd center has a graphical calculus for which the object strands can be though of as ribbon tangles and for which the usual evaluation map is invariant under isotopy of ribbon tangles [2, Sec. 2.3].

We will represent objects (X, c) of the Drinfel'd center with green strands and we will depict the half-bradining as follows:



. (3.16)

3.3 From Modules to the Drinfel'd center and Canonical Restriction

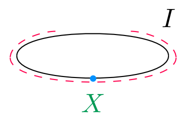
Before understanding structural aspects associated to the modules of the tube category, it is both enlightening and of extreme convenience, to first grasp the connection between the representation category, or the category of modules of $\mathcal{C}$, a spherical fusion category $\mathcal{C}$ and its corresponding tube category $\mathcal{T}(S^1)$. This connection is best expressed by the following lemma:

Lemma 3.9. *There exists a canonical essentially surjective faithful functor:*

$$\Psi \colon \mathcal{C} \longrightarrow \mathrm{Tube}_{\mathcal{C}}(S^1) \tag{3.17}$$

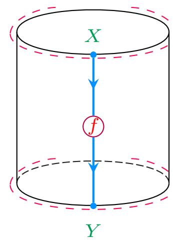
Proof. Let $X \in \mathcal{C}$, pick an open interval $I \subset S^1$. Let $\Psi(X)$ be the following object in $\mathcal{T}(S^1)$:

$$\Psi(X) \quad :=$$



where the single X-labeled point is chosen to be the middle of I, whose embedding in S^1 is indicated by the red dashed outlines. Let $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$, be an arbitrary morphism in $\mathcal{C}$, then Ψ acts as follows on it:

$$\Psi(X \xrightarrow{f} Y) \quad :=$$

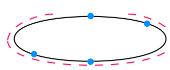


To prove that Ψ is essentially surjective we must, for every object $\{V_b\}$ in the Tube category i.e. a labeled circle, explicitly find the object X in $\mathcal{C}$ and the isomorphism in $\mathrm{Tube}_{\mathcal{C}}(S^1)$, such that:

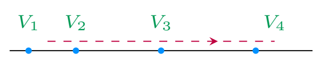
$$\Psi(X) \simeq \{V_b\} \tag{3.18}$$

Consider an arbitrary object:

$$\{V_b\} \quad = \quad$$

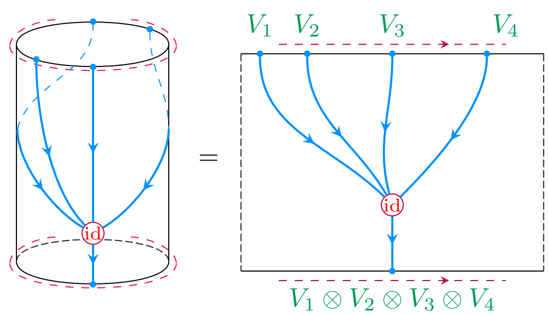


$$=$$



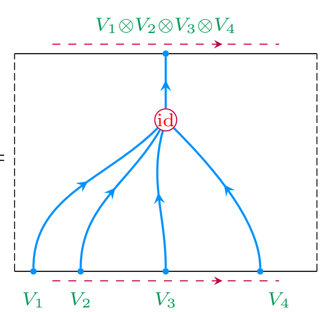
then the morphism depicted below, which we will denote Γ, in $\mathrm{Tube}_{\mathcal{C}}(S^1)$ provides a candidate isomorphism from $\{V_b\}$ to $\Psi(V_1 \otimes V_2 \otimes V_3 \otimes V_4)$:

$$\Gamma \quad := \quad$$

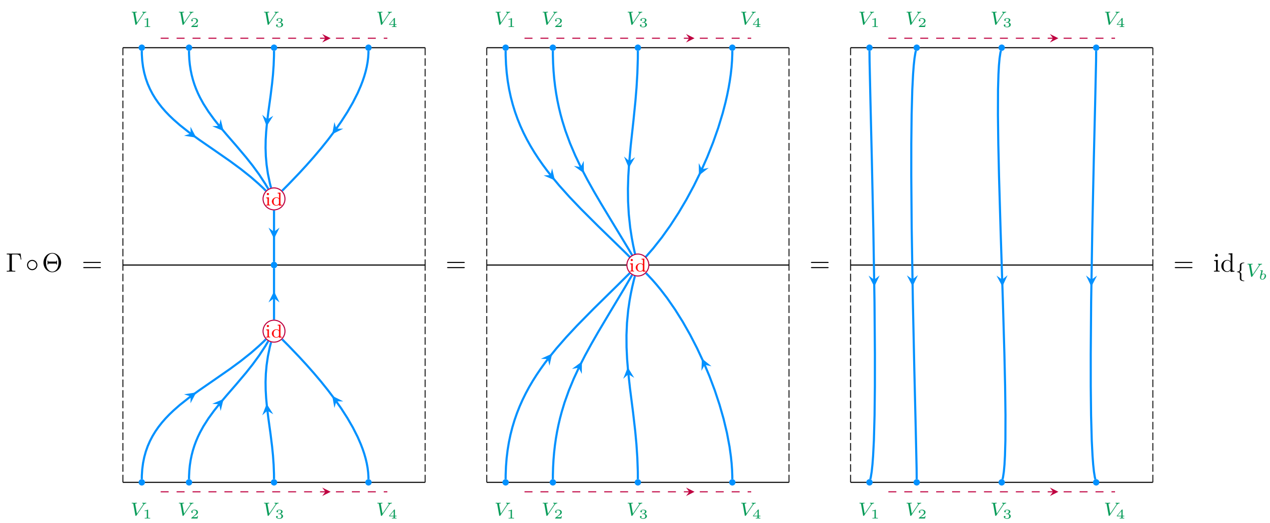


to check that this is in fact an isomorphism we check that the following morphism:

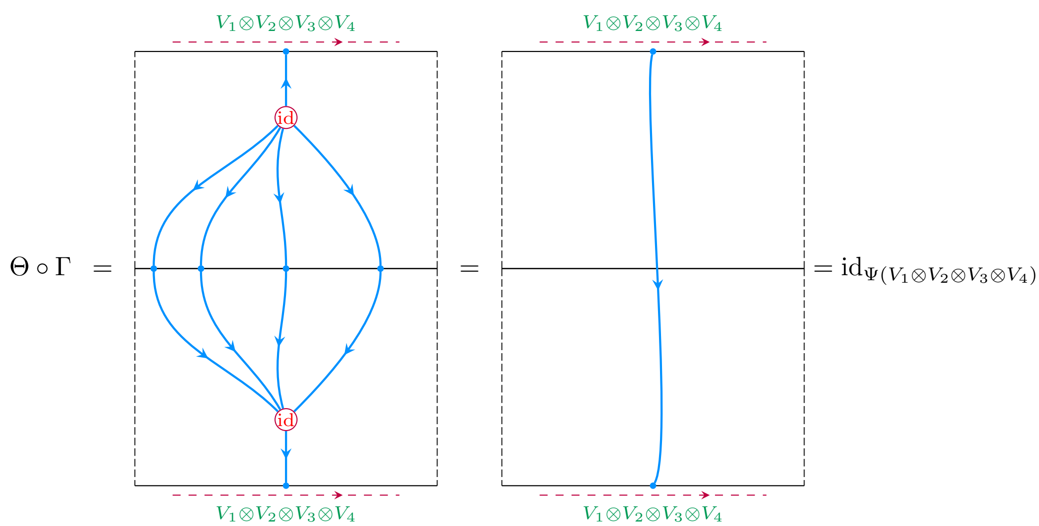
$$\Theta :=$$



is in fact an inverse to Γ. So we compose them:



and:

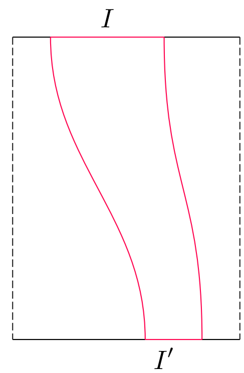


Therefore, Γ and Θ are mutual inverses implying then that Ψ is an essentially surjective functor.

To prove that Ψ is a canonical functor we must show that, up to natural isomorphism, all choices of interval are as good as any another. In more rigorous terms, we have that:

$$\Psi_I \cong \Psi_{I'}. \tag{3.19}$$

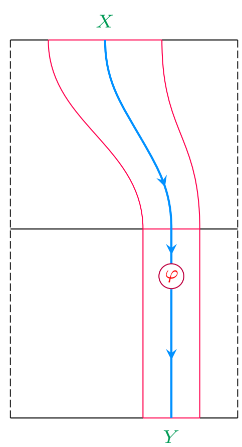
To prove it, let us consider a pair of arbitrary intervals $I, I' \subset S^1$, then we can define a natural transformation by preforming an isotopy that carries I to I', as the drawing below exemplifies:

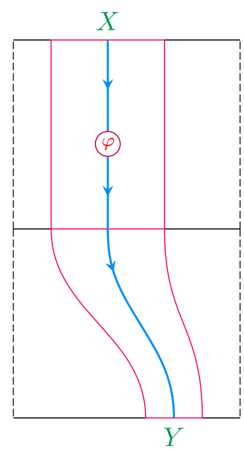


So that, for every $X \in \mathcal{C}$, we define:

$$\eta_X \colon \Psi_I(X) \longrightarrow \Psi_{I'}(X) \tag{3.20}$$

to be the tube that witnesses the isotopy defined by the picture above. Moreover, for every X, η_X is indeed an isomorphism, with inverse defined by the conjugate or opposite tube obtained via vertical reflection. Leaving only the naturality to be checked. This is done by inspection, as it is a consequence of the following equality for every $\varphi \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$:

$\Psi_{I'}(\varphi) \circ \eta_X \quad =$ 

$=$ 

$= \quad \eta_Y \circ \Psi_I(\varphi)$

which finishes the proof. $\square$

This result is extremely important for our aspirations. It tells us that there is a way of studying $\mathrm{Mod}_{\mathcal{T}(S^1)}$ by first studying $\mathrm{Mod}_{\mathcal{C}}$, the reason being that for any two open intervals I, I' and given a module $F \colon \mathrm{Tube}_{\mathcal{C}}(S^1) \longrightarrow \mathrm{Vect}$, there is a natural isomorphism:

$$F \circ \Psi_I \cong F \circ \Psi_{I'}, \tag{3.21}$$

defined by:

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left=50, "\Psi_I"] \arrow[r, bend right=50, "\Psi_{I'}"'] \arrow[r, phantom, "\Downarrow\eta"] & \mathrm{Tube}_{\mathcal{C}}(S^1) \arrow[r, "F"] & \mathrm{Vect} \end{tikzcd} \tag{3.22}$$

where we consider $\mathrm{id}_F \bullet \eta$ horizontal composition of natural transformations. From Appendix ??, we have that, given an arbitrary functor $F \colon \mathcal{C} \longrightarrow \mathrm{Vect}$, there exists an $X \in \mathcal{C}$ such that:

$$F \simeq \mathrm{Hom}_{\mathcal{C}}(R, -). \tag{3.23}$$

This informs us that for any representation $F\colon\ \mathrm{Tube}_{\mathcal{C}}(S^1) \xrightarrow{F} \mathrm{Vect}$ there exists an object $R \in \mathcal{C}$ such that, for all objects $\{V_b\} \in \mathcal{T}(S^1)$ where $\{V_b\} \simeq \Psi(V)$ for some $V \in \mathcal{C}$, we have a vector space isomorphism:

$$F(\{V_b\}) \simeq \mathrm{Hom}_{\mathcal{C}}(R, V) \tag{3.24}$$

and such that, given a pair of objects $\{V_b\}, \{W_c\} \in \mathcal{T}(S^1)$ with $\{V_b\} \simeq \Psi(V)$ and $\{W_c\} \simeq \Psi(W)$ and given a morphism $\Gamma\colon \{V_b\} \to \{W_c\}$ in the image of Ψ i.e. there exists $\varphi \in \mathrm{Hom}_{\mathcal{C}}(V, W)$ such that $\Psi(\varphi) = \Gamma$, then we have that:

$$F(\Gamma\colon \{V_b\} \to \{W_c\})\colon\ \mathrm{Hom}_{\mathcal{C}}(R, V) \xrightarrow{\varphi\circ -} \mathrm{Hom}_{\mathcal{C}}(R, W) \tag{3.25}$$
$$\psi \longmapsto \quad \varphi \circ \psi \tag{3.26}$$

We sum all of this up in the following result:

Proposition 3.10. *Given a spherical fusion category C, let $\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$ be the full subcategory of functors which, upon restriction to C, are represented::*

$$\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)} := \{F \in \mathrm{Mod}_{\mathcal{T}(S^1)} :\ F \circ \Psi = \mathrm{Hom}_{\mathcal{C}}(R, -),\ \textit{for some } R \in \mathcal{C}\}. \tag{3.27}$$

In other words, the category of functors F which, upon precomposition with the inclusion Ψ, are strictly equal to $\mathrm{Hom}(R, -)$ for some R in $\mathcal{C}$. We have the following equivalence:

$$\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)} \simeq \mathrm{Mod}_{\mathcal{T}(S^1)}\,. \tag{3.28}$$

Proof. Consider the inclusion functor:

$$\iota\colon \overline{\mathrm{Mod}}_{\mathcal{T}(S^1)} \longrightarrow \mathrm{Mod}_{\mathcal{T}(S^1)}\,.$$

Since $\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$ is a full subcategory it implies ι is a fully faithful functor, making the result depend exclusively on the proof that ι is a essentially surjective. Essential surjectivity comes from the fact that all functors $\mathcal{C} \longrightarrow \mathrm{Vect}$ are naturally isomorphic to representable functor: □

To ease the burden of having to make highly non-canonical choices, and in light of the equivalence of Proposition 9.1, we choose henceforward to work with $\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$ rather than $\mathrm{Mod}_{\mathcal{T}(S^1)}$ without any loss of generality. Taking this new point of view, a natural question ensues: Given a representation of $\mathcal{C}$:

$$\hat{F}\colon \mathcal{C} \longrightarrow \mathrm{Vect},$$

can we build a unique representation

$$F\colon\ \mathrm{Tube}_{\mathcal{C}}(S^1) \longrightarrow \mathrm{Vect}$$

such that

$$F \circ \Psi \simeq \hat{F}?$$

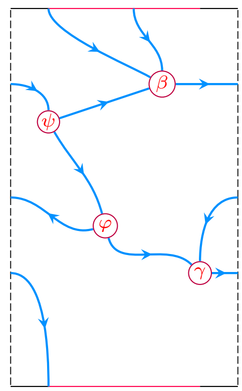
i.e. such that F is a canonical extension of $\hat{F}$? To understand this question we acknowledge, first and foremost, that if such an extension exists, then the action on the objects of $\mathrm{Tube}_{\mathcal{C}}(S^1)$ is completely pinned down, so to speak, by $\hat{F}$'s action on $\mathcal{C}$. It is so, because, if $\bigcirc \simeq \Psi(R)$, then:

$$F(\bigcirc) \simeq F(\Psi(V)) \simeq \hat{F}(V),$$

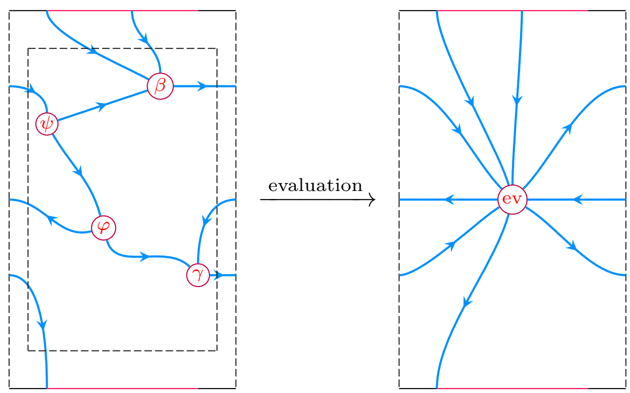
which in turn, from equation 3.23 translates into:

$$F(\bigcirc) \simeq \mathrm{Hom}_{\mathcal{C}}(R, V) \tag{3.29}$$

for a fixed object $R \in \mathcal{C}$. The second, and key, observation we must bring to light has to do with the structure of $\mathrm{Hom}_{\mathcal{T}(S^1)}(-, -)$ morphisms in $\mathrm{Tube}_{\mathcal{C}}(S^1)$. Let us consider an arbitrary string-net:

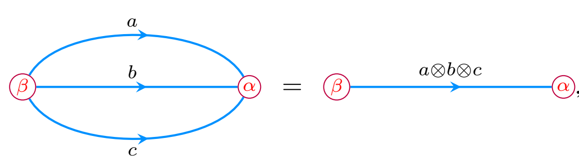


One can then pick a ball containing all the vertices and apply the evaluation map to get a single vertex:

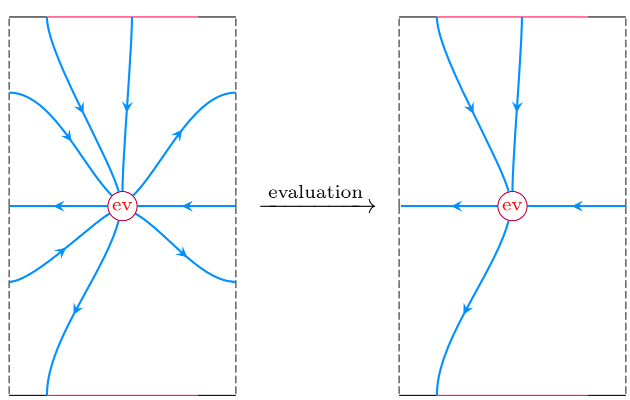


(3.30)

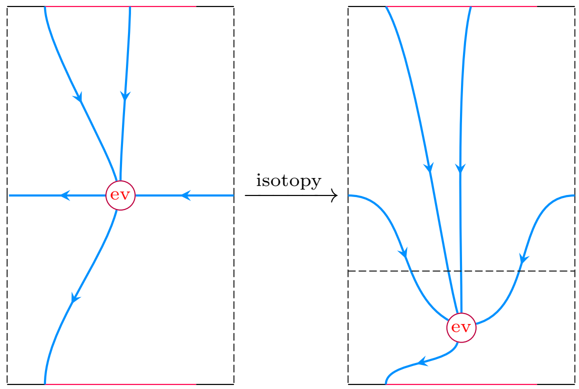
Then, using the edge merge local relation:



we simplify diagram 3.143:

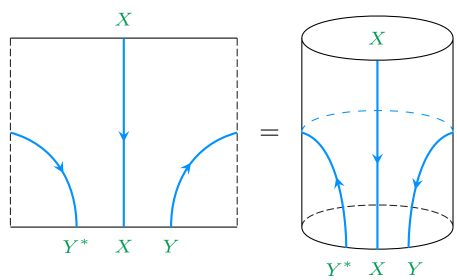


and isotope it:



to obtain a factorization of the original string-net. The two factors are split by elements in the image of the embedding functor Ψ, the bottom one, and elements not included in the image of Ψ, in the top. This is a pivotal observation in our attempt to interpret how a module of the tube category acts on an arbitrary morphism, by using the fact that for morphisms in the image of Ψ, the action, like in the case for objects, is determined by $\hat{F}$, leaving morphisms of the following type:

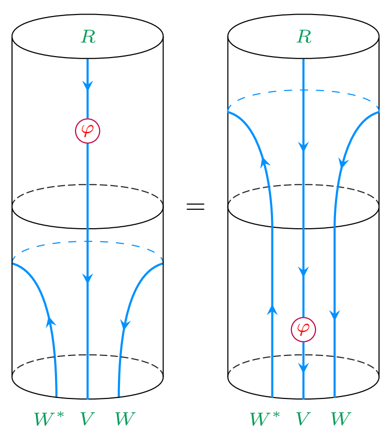
$\beta_{X,Y} =$



(3.31)

to be the remaining rest of data needed to fully define F. Picking up all the pieces, we have that an arbitrary $F \in \overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$ is specified by a representing object $R \in \mathcal{C}$ and by its action on morphisms of the type $\beta_{V,W}$ for all $V, W \in \mathcal{C}$. This set of specifying data for F might still be trimmed down, and to see how let us closely inspect the consequences of the following equality for morphisms in $\mathrm{Tube}_{\mathcal{C}}(S^1)$:

$\beta_{V,W} \circ \Gamma' =$



$= \Gamma \circ \beta_{R,W}$

which holds for all $\varphi \in \mathrm{Hom}_{\mathcal{C}}(R, V)$, where:

$$\Gamma = \Psi(\mathrm{id}_{W^*} \otimes \varphi \otimes \mathrm{id}_W)$$
$$\Gamma' = \Psi(\varphi)$$

which translates into the following algebraic equation:

$$F(\beta_{V,W} \circ \Gamma') = F(\Gamma \circ \beta_{R,W}) \tag{3.32}$$

and since Γ, Γ' are in the image of the inclusion Ψ, we have that:

$$\begin{aligned} F(\Gamma) = F(\Psi(\mathrm{id}_{W^*} \otimes \varphi \otimes \mathrm{id}_W))\colon\ \mathrm{Hom}_{\mathcal{C}}(R,R) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(R, W^* \otimes V \otimes W) \\ \psi &\longmapsto \mathrm{id}_{W^*} \otimes \varphi \otimes \mathrm{id}_W \circ \psi \\ F(\Gamma') = F(\Psi(\varphi))\colon\ \mathrm{Hom}_{\mathcal{C}}(R,R) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(R, V) \\ \psi &\longmapsto \varphi \circ \psi \end{aligned}$$

Equation 3.32 is really a collection of equations with objects in $\mathrm{Hom}_{\mathcal{C}}(R, W^* \otimes R \otimes W)$ parameterized by morphisms in $\mathrm{Hom}_{\mathcal{C}}(R,R)$, and has as a particular case:

$$\begin{aligned} F(\beta_{V,W} \circ \Gamma')(\mathrm{id}_R) &= F(\Gamma \circ \beta_{R,W})(\mathrm{id}_R) \\ \big[F(\beta_{V,W}) \circ F(\Gamma')\big](\mathrm{id}_R) &= \big[F(\Gamma) \circ F(\beta_{R,W})\big](\mathrm{id}_R) \\ F(\beta_{V,W})(\varphi \circ \mathrm{id}_R) &= \mathrm{id}_{W^*} \otimes \varphi \otimes \mathrm{id}_W \circ \big[F(\beta_{R,W})(\mathrm{id}_R)\big] \\ F(\beta_{V,W})(\varphi) &= \mathrm{id}_{W^*} \otimes \varphi \otimes \mathrm{id}_W \circ \big[F(\beta_{R,W})(\mathrm{id}_R)\big], \end{aligned}$$

meaning that the module's action on morphisms of the type $\beta_{V,W}$ is entirely specified by knowing the object $R \in \mathcal{C}$ and how it acts on morphisms of the type $\{\beta_{R,W}\}_{W \in \mathcal{C}}$. Said differently, $F(\beta_{V,W})$ is completely determined by knowing $F(\beta_{R,W}(\mathrm{id}_R))$. In particular, this means we have answered the question:

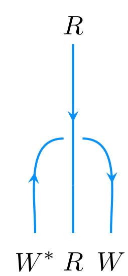
$\boxed{\text{Given a module } \hat{F} \text{ of } \mathcal{C}\text{, how can it be extended to a module } F \text{ of } \mathrm{Tube}_{\mathcal{C}}(S^1)\text{?}}$

The answer comes in the form of specifying the following collection of morphisms in $\mathcal{C}$:

$$\{\tilde{c}_{R,W} := F(\beta_{R,W})(\mathrm{id}_R)\colon R \longrightarrow (W)^* \otimes R \otimes W\}_{W \in \mathcal{C}} \tag{3.33}$$

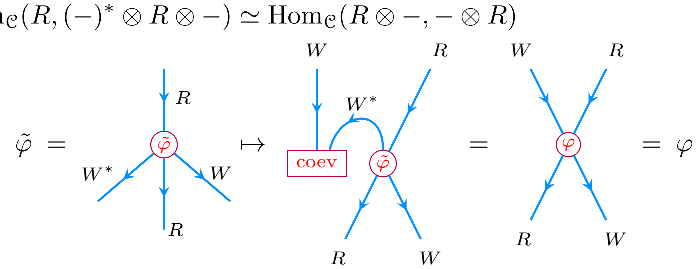
Which we will graphically depict in the following manner:

$\tilde{c}_{R,W} \quad :=$



There is good reason for this choice, as the following results will convey. Moreover, recall that since, in particular, we are working with a pivotal category, we have the following canonical identification called yanking:

$$\mathrm{Hom}_{\mathcal{C}}(R, (-)^* \otimes R \otimes -) \simeq \mathrm{Hom}_{\mathcal{C}}(R \otimes -, - \otimes R) \tag{3.34}$$



(3.35)

We now establish some of the properties of $\tilde{c}_{R,-}$.

Lemma 3.11. *The collection of morphisms* $\{c_{R,W}\}_{W\in\mathcal{C}}$

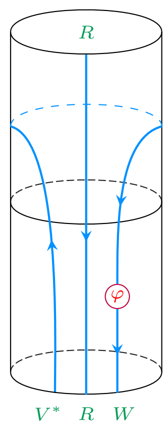
$$c_{R,W} \colon W \otimes R \longrightarrow R \otimes W$$

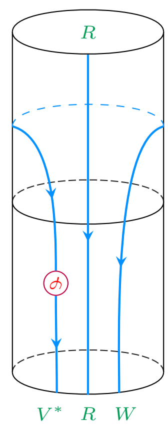
corresponding to the "yanked" image of $\tilde{c}_{R,W}$*, satisfy both* ***naturality conditions*** *and* ***hexagon relations***.

Proof. We begin by proving that $c_{R,-}$ form a natural transformation:

$$-\otimes R \xRightarrow{c_{R,-}} R \otimes -. \tag{3.36}$$

For that, let us consider the following pair of equal morphisms in $\mathrm{Tube}_{\mathcal{C}}(S^1)$:

$\Gamma_\varphi \circ \beta_{R,V} =$ 

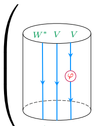
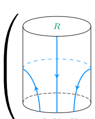
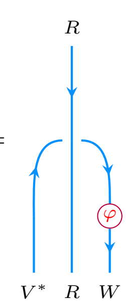
$=$ 

$= \beta_{R,W} \circ \Gamma_{\varphi^*}$

where:

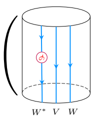
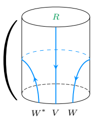
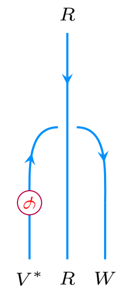
$$\Gamma_\varphi = \Psi(\mathrm{id}_{V^*\otimes R} \otimes \varphi)$$
$$\Gamma_{\varphi^*} = \Psi(\varphi^* \otimes \mathrm{id}_{R\otimes W})$$

Going from left to right, the equivalence is constructed by sliding the coupon on the right, $V \xrightarrow{\varphi} W$, to its dual, $W^* \xrightarrow{\varphi^*} V^*$, on the left. We make use of the notation of rotating coupons by π radians counterclockwise in order to represent the dual. Translating this equality, via the module $F = \mathrm{Hom}_{\mathcal{C}}(R, -)$, we have the following collection of equations for all $V, W \in \mathcal{C}$ and for all $\varphi \in \mathrm{Hom}_{\mathcal{C}}(V, W)$:

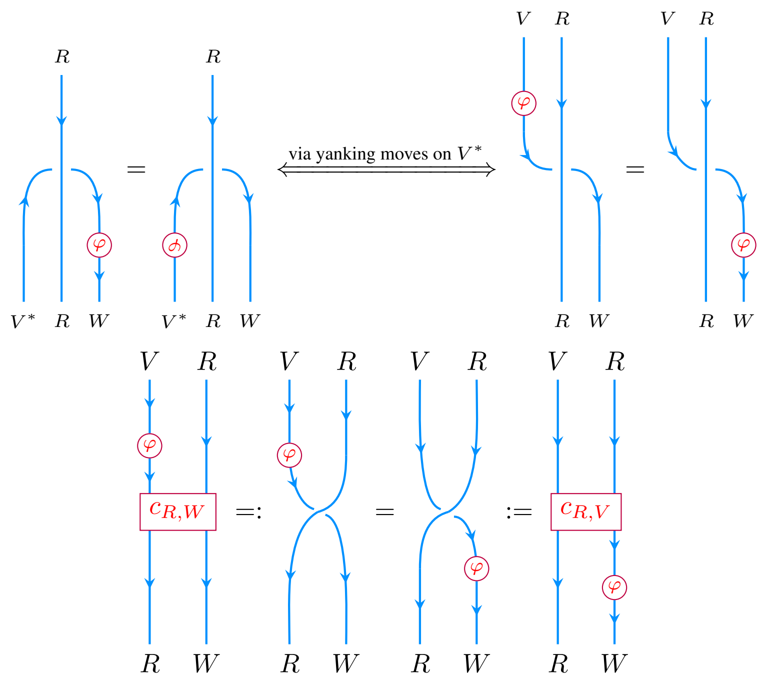
$F\Bigg($  $\Bigg) \circ F\Bigg($  $\Bigg)(\mathrm{id}_R) = F(\Gamma_\varphi) \circ F(\beta_{R,V})(\mathrm{id}_R) = (\mathrm{id}_{V^*\otimes R} \otimes \varphi) \circ \tilde{c}_{R,V} =$ 

(3.37)

on the other hand:

$F\Bigg($  $\Bigg) \circ F\Bigg($  $\Bigg)(\mathrm{id}_R) \;=\; F(\Gamma_\varphi)\circ F(\beta_{R,V})(\mathrm{id}_R) = (\varphi^* \otimes \mathrm{id}_{R\otimes W})\circ \tilde{c}_{R,W} \;=$  (3.38)

Putting together both cases we get the following equation:



Which we can readily sum up in the commutativity of the following diagram:

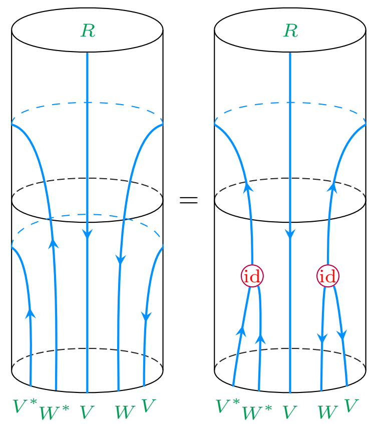
$$\begin{tikzcd}
V\otimes R \arrow[r, "c_{R,V}"] \arrow[d, "\varphi\otimes\mathrm{id}_R"'] & R\otimes V \arrow[d, "\mathrm{id}_R\otimes\varphi"] \\
W\otimes R \arrow[r, "c_{R,W}"'] & R\otimes W
\end{tikzcd} \tag{3.39}$$

for all $\varphi \in \mathrm{Hom}_{\mathcal{C}}(V,W)$.

We now proceed to prove the hexagon relations, so let us first remember what they are:

$$c_{R,V\otimes W} = (c_{R,V}\otimes \mathrm{id}_W)\circ(\mathrm{id}_V\otimes c_{R,W}). \tag{3.40}$$

We apply a method similar to one used for the naturality. For that, we start with the following equality of morphims in $\mathrm{Tube}_{\mathcal{C}}(S^1)$:



$$\beta_{W^*\otimes R\otimes W,V}\circ\beta_{R,W} \;=\; \cdots \;=\; \cdots \;=\; \Gamma\circ\beta_{V,W}, \tag{3.41}$$

where:

$$\Gamma = \Psi(\mathrm{id}_{V^*\otimes W^*\otimes R\otimes W\otimes V}).$$

Evaluating both sides on the identity of R, and starting with the left-hand side gives:

$$[F(\beta_{W^*\otimes R\otimes W,V})\circ F(\beta_{R,W})](\mathrm{id}_R) = F(\beta_{W^*\otimes R\otimes W,V})(\tilde{c}_{R,V}) = (\mathrm{id}_{W^*}\ \otimes\tilde{c}_{R,W}\otimes\mathrm{id}_W)\circ\tilde{c}_{R,W}. \tag{3.42}$$

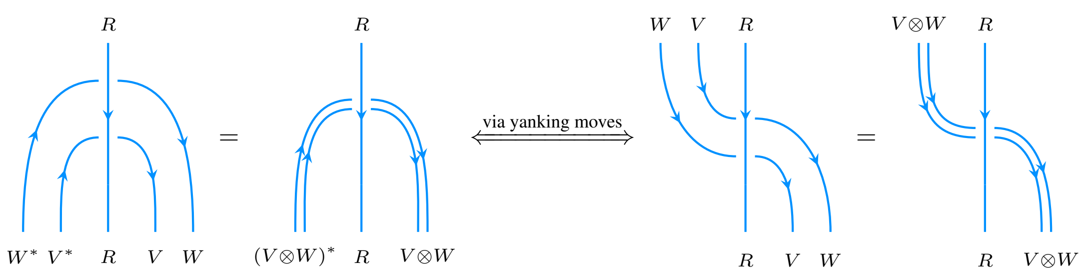
From the diagram on the right-hand side, we get:

$$F(\Gamma)\circ F(\beta_{R,V\otimes W})(\mathrm{id}_R) = F(\Gamma)(\tilde{c}_{R,V\otimes W}) = \mathrm{id}_{V^*\otimes W^*\otimes R\otimes W\otimes V}\circ\tilde{c}_{R,V\otimes W} = \tilde{c}_{R,V\otimes W}. \tag{3.43}$$

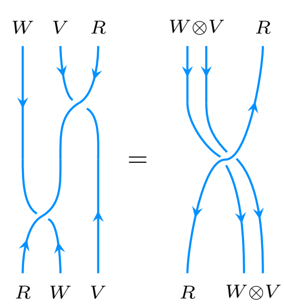
Putting together equations (27) and (28), as a consequence of (26), we get the following relation:

$$[\mathrm{id}_{W^*}\ \otimes\tilde{c}_{R,W}\otimes\mathrm{id}_W]\circ\tilde{c}_{R,W} \;=\; \tilde{c}_{R,V\otimes W} \tag{3.44}$$

which graphically translates into:



Now, translating the two "yanked" equal morphisms on the right into more "familiar" string diagrams, we get:



$$(c_{R,W}\otimes\mathrm{id}_V)\circ(\mathrm{id}_W\otimes c_{R,V}) \;=\; \cdots \;=\; \cdots \;=\; c_{R,W\otimes V} \tag{3.45}$$

concluding the proof of the result. $\square$

We still need to check that $c_{R,W}$ is, an isomorphism for all $W \in \mathcal{C}$, in order to ensure that we have successfully produced a half-braiding. The next result proves that this is the case:

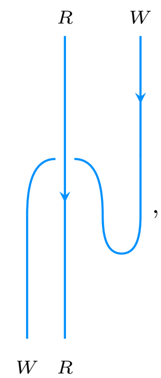
Lemma 3.12. *(Isomorphism) Let F be a module of the tube category and let:*

$$c_{R,W} := \mathrm{ev}_W \circ F(\beta_{R,W}) \tag{3.46}$$

be our candidate for a half-braiding. We define:

$$e \colon W \otimes R \longrightarrow R \otimes W, \tag{3.47}$$

to be the morphism in $\mathcal{C}$ defined as follows:

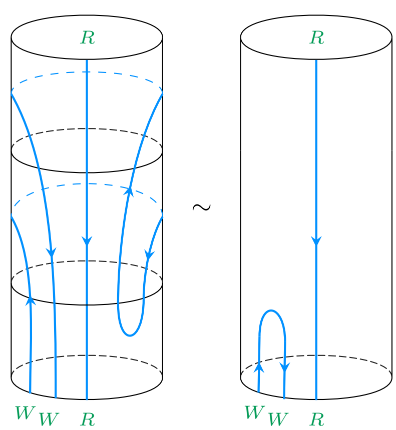


, (3.48)

then we have that:

$$e \circ c_{R,W} = \mathrm{id}_{R\otimes W} \quad \textit{and} \quad c_{R,W} \circ e = \mathrm{id}_{W\otimes R}\,. \tag{3.49}$$

Proof. We begin with the proof with a pair of equivalent string-nets on the cylinder:



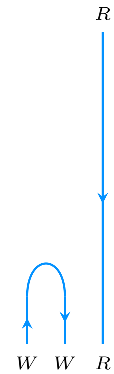
(3.50)

which we pin down algebraically as follows:

$$\Psi\big((\mathrm{id}_{W^*\otimes W\otimes R} \otimes \mathrm{ev}_W\big) \circ \big(\beta_{W\otimes R\otimes W^*,W}\big) \circ \big(\beta_{R,W^*}\big) = \Psi\big(\mathrm{coev}^W \otimes \mathrm{id}_R\big) \tag{3.51}$$

Importantly, we can use the machinery of the module F in order "restate" the equation above in $\mathcal{C}$. To do so, we apply F to both sides and compare how the identity morphism of R is acted upon using first the right and then the left-hand sides of equation (12). We begin with the right-hand side:

$$F\big(\mathrm{coev}^W \otimes \mathrm{id}_R\big)(\mathrm{id}_R) \;=\; (\mathrm{coev}^W \otimes \mathrm{id}_R) \;= \tag{3.52}$$



while taking the morphims on the left-hand side we get:

$$F(\Psi\big((\mathrm{id}_{W^*\otimes W\otimes R} \otimes \mathrm{ev}_W) \circ (\beta_{W\otimes R\otimes W^*,W}) \circ (\beta_{R,W^*})\big)(\mathrm{id}_R) = F\big(\Psi(\mathrm{id}_{W^*\otimes W\otimes R} \otimes \mathrm{ev}_W)\big) \circ F(\beta_{W\otimes R\otimes W^*,W}) \circ F(\beta_{R,W^*})\big)(\mathrm{id}_R)$$

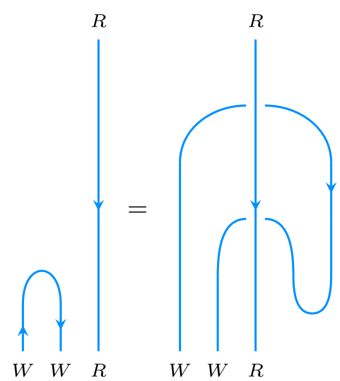
as a direct consequence of F being a functor. Expanding the right-hand side, we have:

$$F\big(\Psi(\mathrm{id}_{W^*\otimes W\otimes R} \otimes \mathrm{ev}_W)\big) \circ F(\beta_{W\otimes R\otimes W^*,W})(\tilde{c}_{R,W^*}) \;=\; F\big(\Psi(\mathrm{id}_{W^*\otimes W\otimes R} \otimes \mathrm{ev}_W)\big)\big((\mathrm{id}_{W^*} \otimes \tilde{c}_{R,W^*} \otimes \mathrm{id}_W) \circ \tilde{c}_{R,W}\big) \tag{3.53}$$

and hence, combining equation (13) with the two equations above, we get the following equality:

$$\mathrm{coev}^W \otimes \mathrm{id}_R \;=\; \mathrm{id}_{W^*\otimes W\otimes R} \otimes \mathrm{ev}_W) \circ (\mathrm{id}_{W^*} \otimes \tilde{c}_{R,W^*} \otimes \mathrm{id}_W) \circ \tilde{c}_{R,W}\big) \tag{3.54}$$

graphically depicted as:

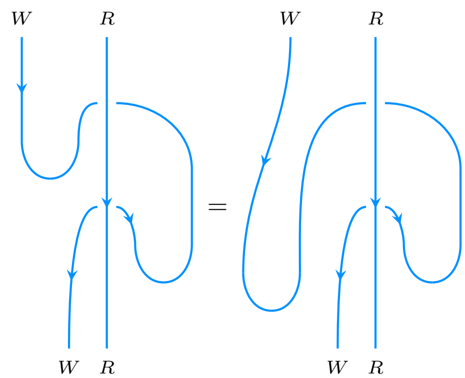


(3.55)

providing the necessary identity to prove our goal result:

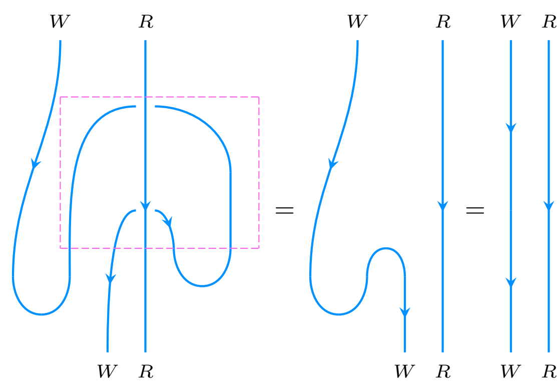
$$e \circ c_{R,W} = \mathrm{id}_{W\otimes R} \tag{3.56}$$

To check this we argue diagrammatically:

$e \circ c_{R,W} \;=$ 

(3.57)

now we apply the previous result to the picture on the right as follows:



(3.58)

proving that e is the left-inverse of $c_{R,W}$. The proof for the mirror identity is done similarly. $\square$

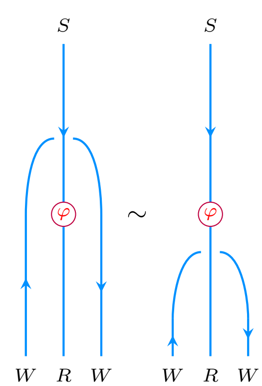
Next, we would like to ensure that given a morphism between a pair of arbitrary modules F, G in $\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$ i.e. a natural transformation:

$$\eta \colon F \Longrightarrow G$$

we can uniquely "reassemble" a morphism:

$$\varphi \colon (S, d) \longrightarrow (R, c)$$

where R and S are the representing objects of F and G, respectively, and c and d are the candidates for half-braidings we have been introducing throughout this section. Moreover, in order for φ to be a morphism in $\mathcal{Z}(\mathcal{C})$, we need it to satisfy the following "unyanked" relation:

$(\mathrm{id}_{W^*} \otimes \varphi \otimes \mathrm{id}_W) \circ \tilde{d}_{S,W} \;=$ 

$= \;\tilde{c}_{R,W} \circ \varphi$ (3.59)

The Yoneda lemma and Yoneda embedding suggest there is a very natural candidate. Following the proof of the aforementioned result, we have that, since η is a natural transformation between representable functors, when canonically restricted to $\mathcal{C}$, all components η are uniquely determined by:

$$\varphi := \eta_{\Psi(R)}(\mathrm{id}_R) \in \mathrm{Hom}_{\mathcal{C}}(S, R), \tag{3.60}$$

in particular, they are given by pre-composition:

$$\eta_{\Psi(R)}(\psi) = \psi \circ \varphi. \tag{3.61}$$

We need to check that this morphism satisfies the relations defined by equation 3.59. We note that, writing the naturality of η for the morphism $\beta_{R,Z}$, we get the following equation:

$$\eta_{\Psi(Z^* \otimes R \otimes Z)}(F(\beta_{R,Z})) = G(\beta_{R,Z})(\eta_{\Psi(R)}). \tag{3.62}$$

Evaluating both sides at id_R gives:

$$\eta_{\Psi(Z^* \otimes R \otimes Z)}(\tilde{c}_{R,Z}) = G(\beta_{R,Z})(\varphi), \tag{3.63}$$

recall that we defined $\varphi = \eta_{\Psi(R)}(\mathrm{id}_R)$. From equation 3.43 we get that the right-hand side is

$$\mathrm{id}_{W^* \otimes R \otimes W} \circ \tilde{d}_{S,W}. \tag{3.64}$$

Using the Yoneda lemma for a pair of representable functors, we get that the left-hand side can be rewritten as

$$\tilde{c}_{R,W} \circ \eta_{\Psi(R)}(\mathrm{id}_R) = \tilde{c}_{R,W} \circ \varphi. \tag{3.65}$$

Combining equation 3.64 and 3.65, we get back equation 3.59.

We sum all of the results above in an adaptation of a theorem due to [9].

Theorem 3.13. *("De-tubing Theorem") Let $\mathcal{C}$ be a spherical fusion category. We define a functor:*

$$D\colon \overline{\mathrm{Mod}}_{\mathcal{T}(S^1)} \longrightarrow \mathcal{Z}(\mathcal{C}), \tag{3.66}$$

by mapping every functor $F \in \overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$ to the pair $(R, c) \in \mathcal{Z}(\mathcal{C})$, where R is the representing object of the canonical restriction of F and c is the natural transformation

$$c\colon\ - \otimes R \Longrightarrow R \otimes -, \tag{3.67}$$

defined by $c_{R,W} := (\mathrm{ev}_W \otimes \mathrm{id}_{R \otimes W}) \circ F(\beta_{R,W})(\mathrm{id}_R)$. Given a morphism

$$\eta\colon F \Longrightarrow G, \tag{3.68}$$

of modules in $\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$, such that $D(G) = (S, d)$. Then, we define $D(\eta)$ as follows:

$$D(\eta) := \eta_{\Psi(R)}(\mathrm{id}_R) \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(S, R). \tag{3.69}$$

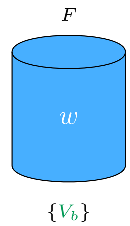
Proof. By putting together Lemma 3.11 and Lemma 3.12, we ensure that $C_{R,W}$ is in fact a half-braiding and hence guaranteeing that (R, c) is in fact an object of $\mathcal{Z}(\mathcal{C})$. The discussion above, culminating in Equation **??** ensures that $D(\eta) := \eta_{\Psi(R)}(\mathrm{id}_R) \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(S, R)$ and functoriality is direct from the construction. □

3.3.1 Braided Monoidal Structure

Before we can prove that $\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$ and $\mathcal{Z}(\mathcal{C})$ are equivalent as braided monoidal categories, we must first associate to $\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$ a braided monoidal structure. Our strategy is to define it for $\mathrm{Mod}_{\mathcal{T}(S^1)}$ and to then canonically restrict it to $\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$. The first challenge is to equip $\mathrm{Mod}_{\mathcal{T}(S^1)}$ with a monoidal structure, in other words a linear functor:

$$\diamond\colon\ \mathrm{Mod}_{\mathcal{T}(S^1)} \times \mathrm{Mod}_{\mathcal{T}(S^1)} \longrightarrow \mathrm{Mod}_{\mathcal{T}(S^1)} \tag{3.70}$$

with a compatible associator and left/right unitors. Before delving into the definition we first introduce a graphical standard that will help us keep track of data associated to arbitrary modules throughout the rest of our discussion. We will denote a vector $w \in F(\{V_b\})$, where $F \in \mathrm{Mod}_{\mathcal{T}(S^1)}$, as follows:



Definition 3.14. Let $F, G \in \mathrm{Mod}_{\mathcal{T}(S^1)}$, then we define the following module:

$$F \diamond G := (F \boxtimes G) \bigotimes_{\mathrm{Tube}_{\mathcal{C}}(S^1 \sqcup S^1)} \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\quad) \tag{3.71}$$

which we call the tensor product of F and G.

In order to break down this definition, let us examine how $F \diamond G$ acts on objects. Let $\{V_b\}$ be an object in $\mathrm{Tube}_{\mathcal{C}}(S^1)$, then we have that:

$$F\ \times\ G(\{V_b\}) := \left[\bigoplus_{\{W_c\},\{U_d\}} (F \boxtimes G)(\{W_c\}, \{U_d\}) \times \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}(\quad; \{W_c\}^\vee, \{U_d\}^\vee, \{V_b\}) \right] \Big/ \sim \tag{3.72}$$

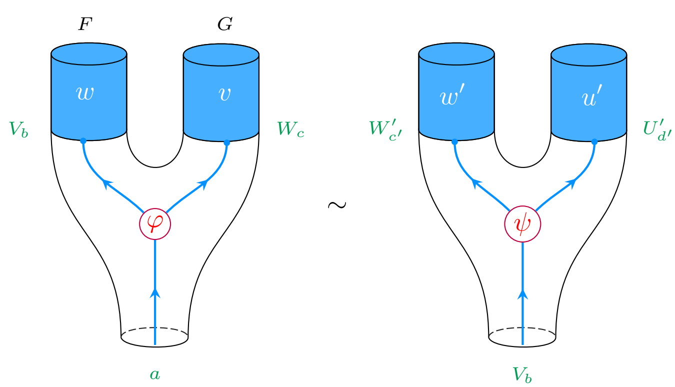
where $\sim$ is defined such that given:

$$\Gamma_1\colon \{W_c\} \longrightarrow \{W'_{c'}\} \ \text{ and } \ \Gamma_2\colon \{U_d\} \longrightarrow, \{U'_{d'}\} \tag{3.73}$$

and given vectors $w \in F(\{W_c\})$ and $u \in G(, \{U_d\})$, we identify:

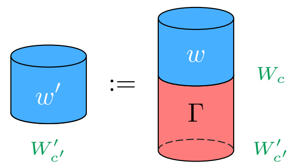
$$w \sim w' = F(\{W_c\} \xrightarrow{\Gamma_1} \{W'_{c'}\})(w) \ \text{ and } \ u \sim u' = (\{U_d\} \xrightarrow{\Gamma_2} \{U'_{d'}\})(u) \tag{3.74}$$

We graphically capture the equivalence relation $\sim$ as follows:



(3.75)

such that



(3.76)

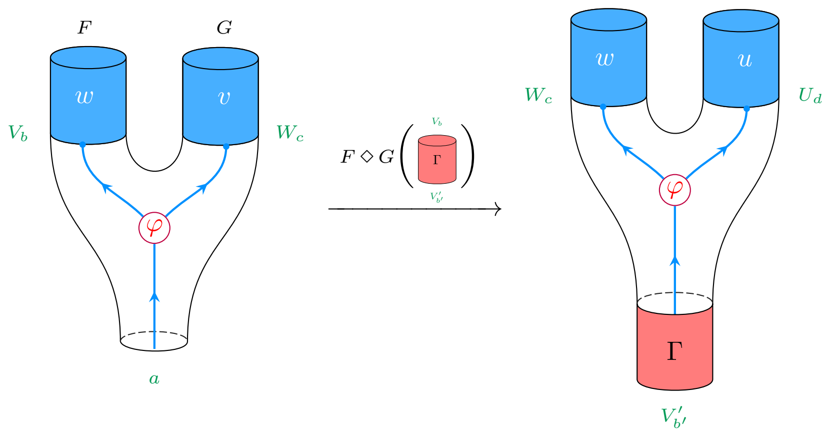
where we introduce a new graphical notation to deal with data associated to arbitrary representations. as a way to graphically keep track of data while doing calculations or other manipulations with the tensor product of modules. To better understand this definition let us look at it through a simple example. Having understood how $F \diamond G$ acts on objects, one is left only with the task of explaining how it acts on morphisms. Given a morphism:

$$\Gamma \colon \{V_b\} \longrightarrow \{V'_{b'}\} \tag{3.77}$$

then we have that:

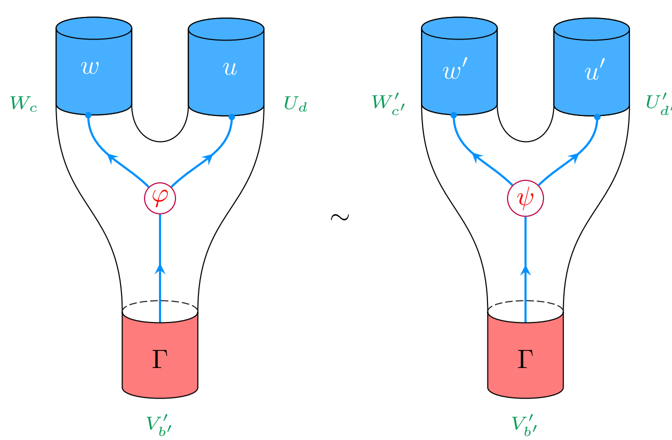
$$(F \diamond G)(\Gamma) \colon (F \diamond G)(\{V_b\}) \longrightarrow (F \diamond G)(\{V'_{b'}\}) \tag{3.78}$$

is defined as the linear map obtained via gluing Γ to the "waist" of the pair-of-pants:



(3.79)

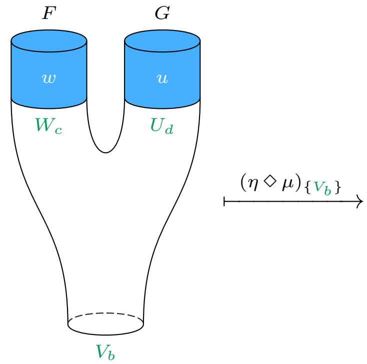
which is a well-defined map since:



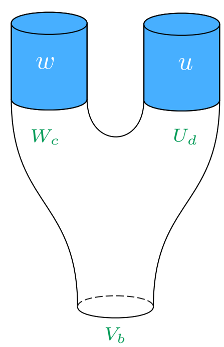
(3.80)

The next step in putting together a tensor structure and ensuring that $\diamond$ is in fact a bifunctor, is to explain how to tensor morphisms of modules i.e. natural transformations between pairs of functors from $\mathrm{Tube}_{\mathcal{C}}(S^1)$ to Vect.

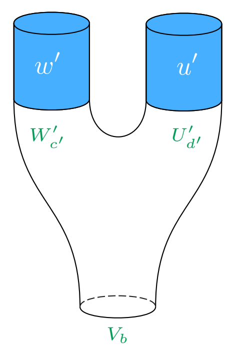
Definition 3.15. Let $F \overset{\eta}{\Longrightarrow} F'$ and $G \overset{\mu}{\Longrightarrow} G'$ be a pair of natural transformations in $\mathrm{Mod}_{\mathcal{T}(S^1)}$, then we define:



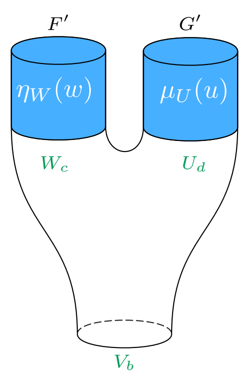
Although, to ensure this is a well-defined linear map we need a little more work. We consider two representative vectors of the same equivalence class in $F \diamond G(V_b)$, which we depict diagrammatically below:



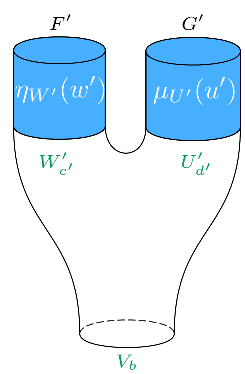
and



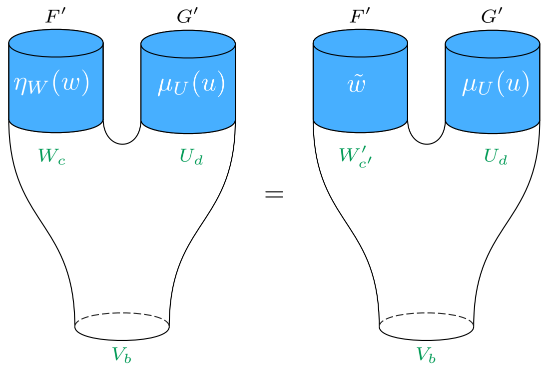
then their respective images under $(\eta \diamond \mu)_{V_b}$ are given by the pair:



and



well-definedness of $(\eta \diamond \mu)_{V_b}$ is summed up by establishing equality between the above two diagrams. Focusing on the left leg, as the case of the right leg follows a completely mirrored story, we have that for the representative on the left, we have the following equality:



where $\tilde{w} := F'(\Gamma)\big(\eta_W(w)\big)$, therefore equality is a consequence of naturality of η and the fact that $w' := F(\Gamma)(w)$:

$$\tilde{w} = F'(\Gamma)\big(\eta_W(w)\big) = \eta_{W'}\big(F(\Gamma)(w)\big) = \eta_{W'}(w')$$

which finishes our inspection At this point, most of the ground-work has been laid to establish the following result:

Theorem 3.16. $\diamond$ *is a bifunctor.*

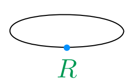
Proof. Above we have shown that $\diamond$ is well-defined on objects and morphisms, leaving only aspects of functoriality to be checked. We need the following checklist to hold:

1. $\big(F \diamond G\big)(\mathrm{id}_{V_b}) = \mathrm{id}_{(F \diamond G)(V_b)}$;
2. $\mathrm{id}_F \diamond \mathrm{id}_G = \mathrm{id}_{F \diamond G}$;
3. $(\eta \circ \eta') \diamond \mu = (\eta \diamond \mathrm{id}) \circ (\eta' \diamond \mu) = (\eta \diamond \mu) \circ (\eta' \diamond \mathrm{id})$;
4. $\eta \diamond (\mu \circ \mu') = (\mathrm{id} \diamond \mu) \circ (\eta \diamond \mu') = (\eta \diamond \mu) \circ (\mathrm{id} \diamond \mu')$

for all $\eta\, \eta'$. This is a routine check and so we skip it. $\square$

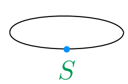
Example 3.17. Let

$$\{V_b\} = \tag{3.81}$$



and

$$\{W_c\} = \tag{3.82}$$



be a pair of objects of $\mathrm{Tube}_{\mathcal{C}}(S^1)$. We then define the follwoing pair of modules:

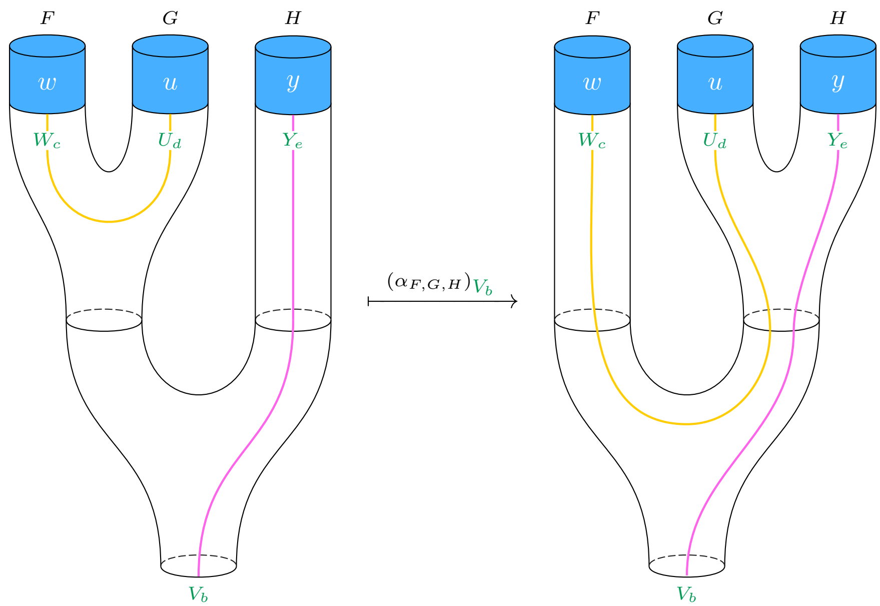
$$F := \mathrm{Hom}_{\mathcal{T}(S^1)}(\{V_b\}, -) \quad \text{and} \quad G := \mathrm{Hom}_{\mathcal{T}(S^1)}(\{W_c\}, -) \tag{3.83}$$

The next step is to introduce a monoidal unit and to define natural isomorphisms corresponding to the associator and the left/right unitor. Let us begin with the associator.

What we are foraging for here, is a set of isomorphisms given for every triple of objects $F, G, H \in \mathrm{Mod}_{\mathcal{T}(S^1)}$ that satisfies the famous pentagon equations. With that goal in mind, fixing a triple $F, G, H \in \mathrm{Mod}_{\mathcal{T}(S^1)}$ of arbitrary modules, we introduce the following linear map:

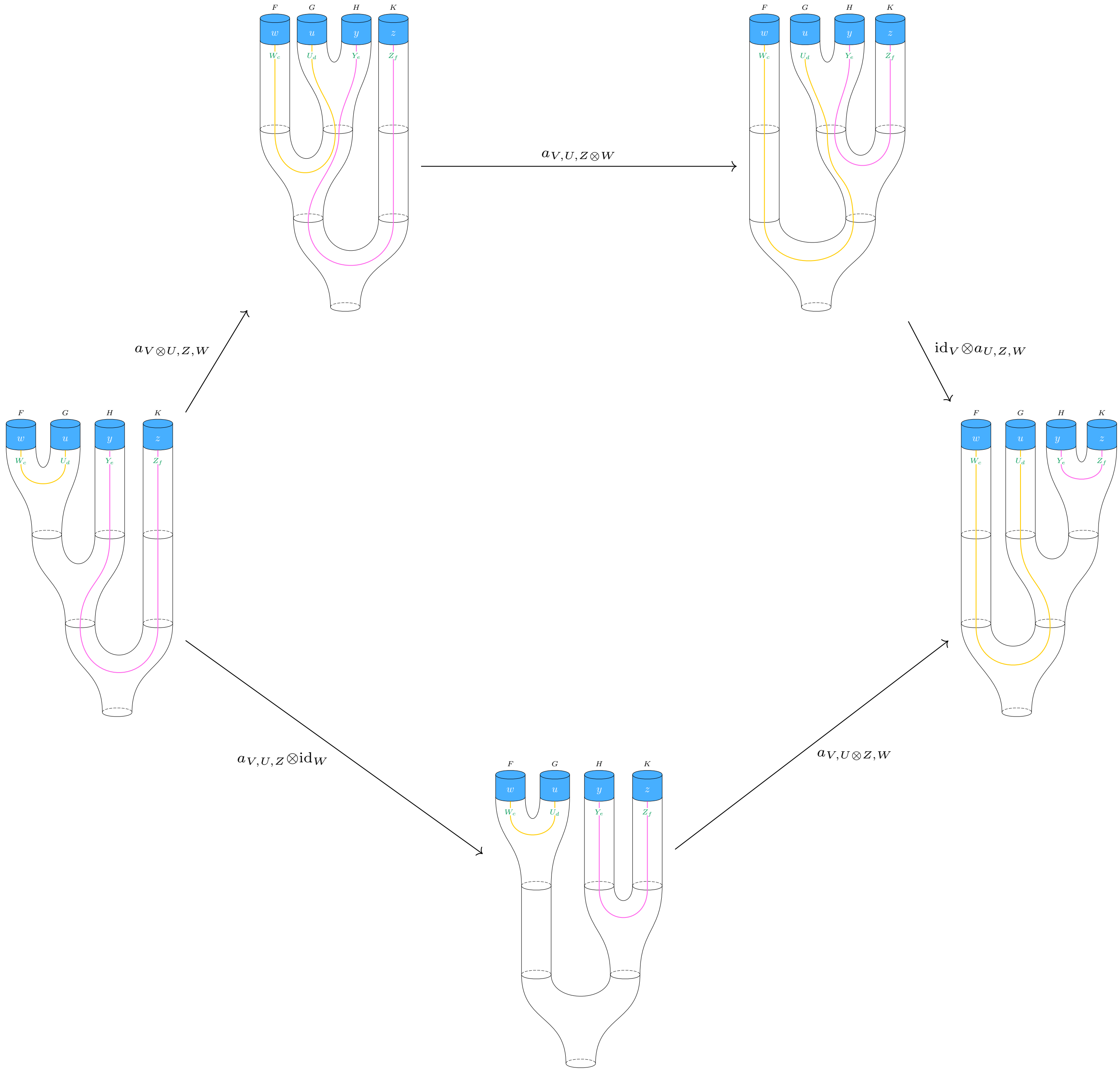
$$\alpha_{F,G,H} \colon (F \diamond G) \diamond H \longrightarrow F \diamond (G \diamond H) \tag{3.84}$$

defined by the following diffeomorphism whose action we trace on a pair of coloured curves:



(3.85)

which is in fact a well-define map and whose naturality is guaranteed by the fact that the operation of gluing a morphism Γ onto the waist, commutes with the application of the diffeomorphism $\alpha_{F,G,H}$. Moreover,the pentagon equations are satisfied, for all objects $F, G, H, K \in \mathrm{Mod}_{\mathcal{T}(S^1)}$, as a consequence of the following diagram:


$a_{V\otimes U,Z,W}$
$a_{V,U,Z\otimes W}$
$\mathrm{id}_V \otimes a_{U,Z,W}$
$a_{V,U,Z}\otimes \mathrm{id}_W$
$a_{V,U\otimes Z,W}$

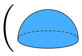
This establishes an associator for $\mathrm{Mod}_{\mathcal{T}(S^1)}$:

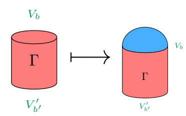
$$\alpha_{F,G,H} \colon (F \diamond G) \diamond H \longrightarrow F \diamond (G \diamond H) \tag{3.86}$$

In order for it to be a tensor product we still need to define a tensor unit.

Definition 3.18. (*Monoidal Unit*) We introduce the following module:

$$\mathbf{1} \colon \ \mathrm{Tube}_{\mathcal{C}}(S^1) \longrightarrow \mathrm{Vect}$$

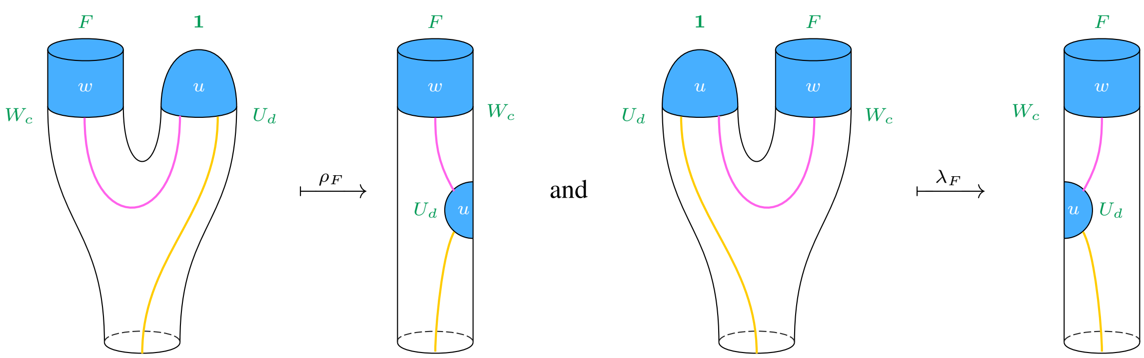
$\{V_b\} \longmapsto \mathcal{Z}^{\mathrm{SN}}_{\mathcal{C}}\Big($  $; V_b\Big)$



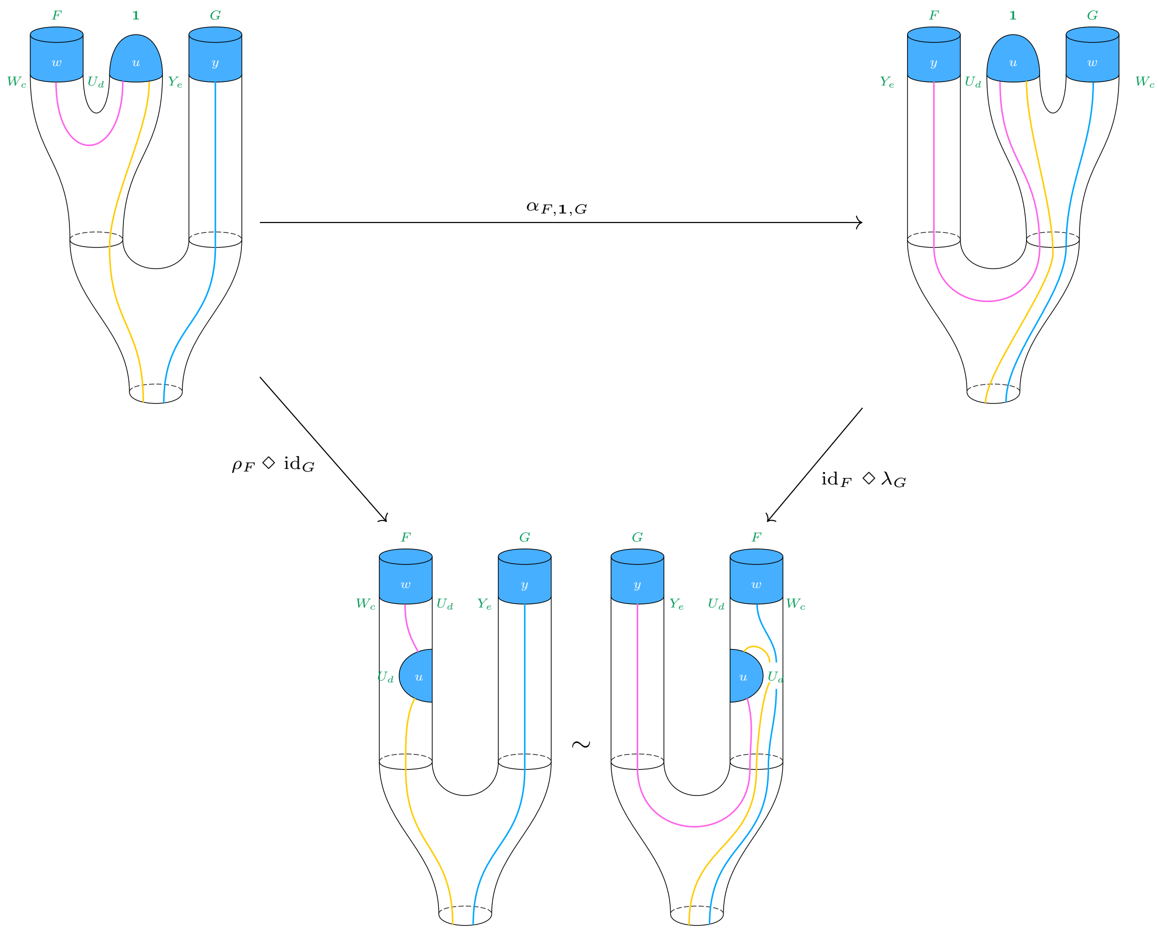
to be the functor that maps a boundary condition to a string-net on a disk with that same boundary condition and such that a morphism Γ is mapped to the linear map that glues a disk on top of Γ, as is depicted in the diagram above. Moreover, we define the following pair of natural isomorphisms:

$$\rho_F \colon (\mathbf{1} \diamond F) \ \longrightarrow F \quad \text{and} \quad \lambda_F \colon (F \diamond \mathbf{1}) \longrightarrow F \tag{3.87}$$

which we define using the following diffeomorphism, represented graphically, as follows:



Here, as usual, we make use of the blue collar notation to represent vectors in the image vector space of the action, by the functor, on the boundary condition indicated either below it or to its side and with the colored curves as an additional source of clarity in expressing the action of the aforementioned diffeomorphism. Moreover, the maps ρ and λ, satisfy, for all $F, G \in \mathrm{Mod}_{\mathcal{T}(S^1)}$, the following commutative diagram:



Where the equivalence in the bottom vertex of the above triangle is given by "sliding" from the left to right leg, as the colored curves, hopefully, convey. Checking that ρ and λ satisfy the triangle relations and, at last, proving that $(\mathbf{1}, \rho, \lambda)$ is a monoidal unit and that $\big(\mathrm{Mod}_{\mathcal{T}(S^1)}, \diamond, \mathbf{1}, \rho, \lambda\big)$ is a tensor category.

Having established $\mathrm{Mod}_{\mathcal{T}(S^1)}$ as a tensor category, one might ask whether there is also a rigid structure.

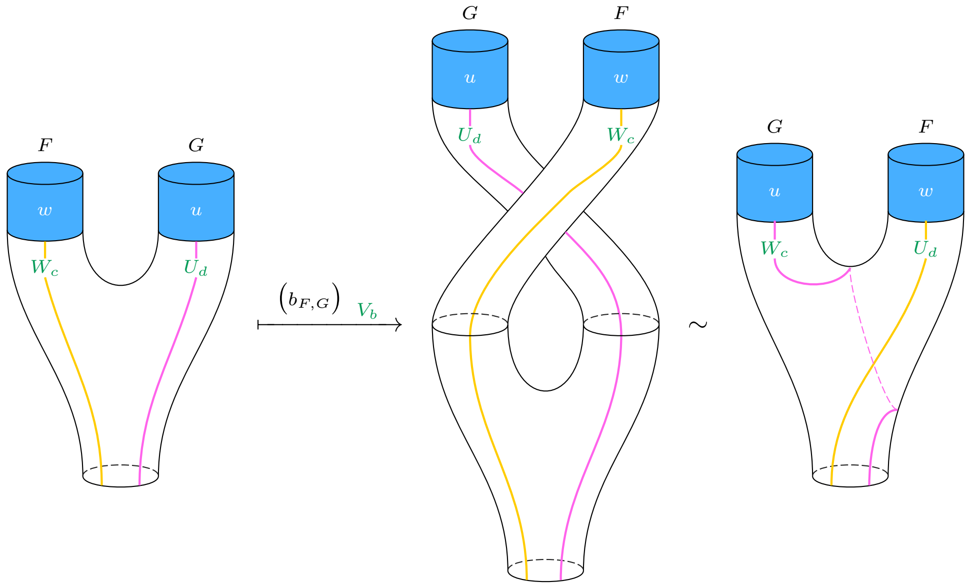
3.4 Braiding Structure

In this subsection we introduce a braiding structure for $\mathrm{Mod}_{\mathrm{Tube}_{\mathcal{C}}(S^1)}$. In order to do so, we must define a collection natural isomorphisms, for all pairs $F, G \in \mathrm{Mod}_{\mathrm{Tube}_{\mathcal{C}\,S^1}}$:

$$\big(b_{F,G}\big)_{V_b} : (F \diamond G)(V_b) \longrightarrow (G \diamond F)(V_b) \tag{3.88}$$

satisfying the hexagon equations. With that aim, we introduce:

Definition 3.19. (*Braiding Structure*) Let F and G be a pair of modules of the tube catgeory. We define the following collection of linear maps:

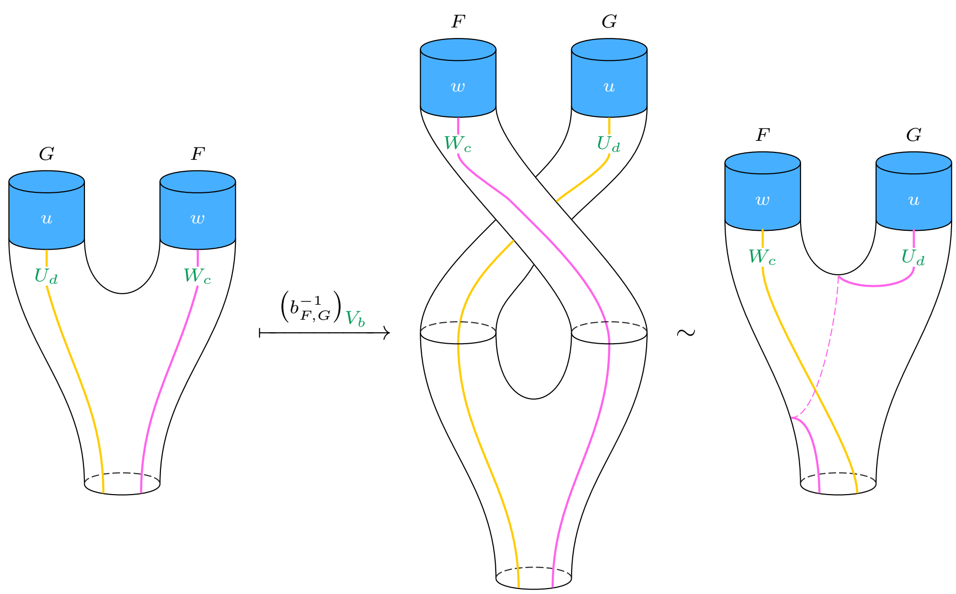


(3.89)

where the relation $\sim$ is defined by the diffeomorphism, relative to the boundary, registered by the pair of coloured curves.

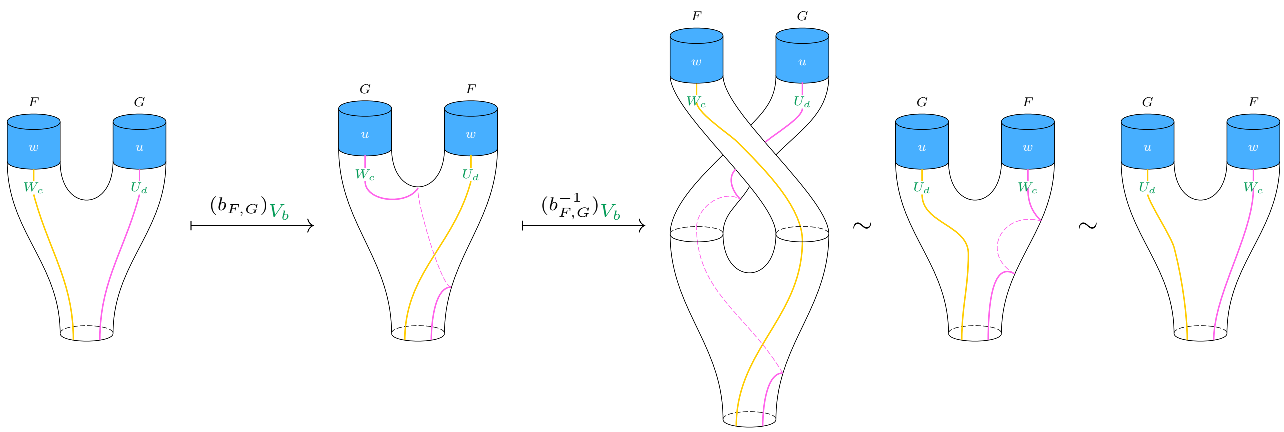
Proposition 3.20. *The collection of linear maps defined by 3.19, form a collection of well-defined, natural isomorphims that satisfy the hexagon equations.*

Proof. Both the well-defined and natural qualities of $b_{F,G}$ come as a direct consequence of the fact that the isotopy used to define it, is applied relative to the boundary. Leaving out only the introduction of an inverse and the proof of the hexagon relations, and we begin with the former. The natural candidate for $b_{F,G}^{-1}$ is the map that swaps the "legs" in the reverse order, rather than passing the left leg in front, it passes the right. Diagrammatically portrayed as:



(3.90)

In order to check this is in fact an isomorphim, we compose both linear maps:



and hence, for all objects V_b:

$$(b_{F,G})_{V_b} \circ (b^{-1}_{F,G})_{V_b} = \mathrm{id}_{(F \diamond G)(V_b)} \tag{3.91}$$

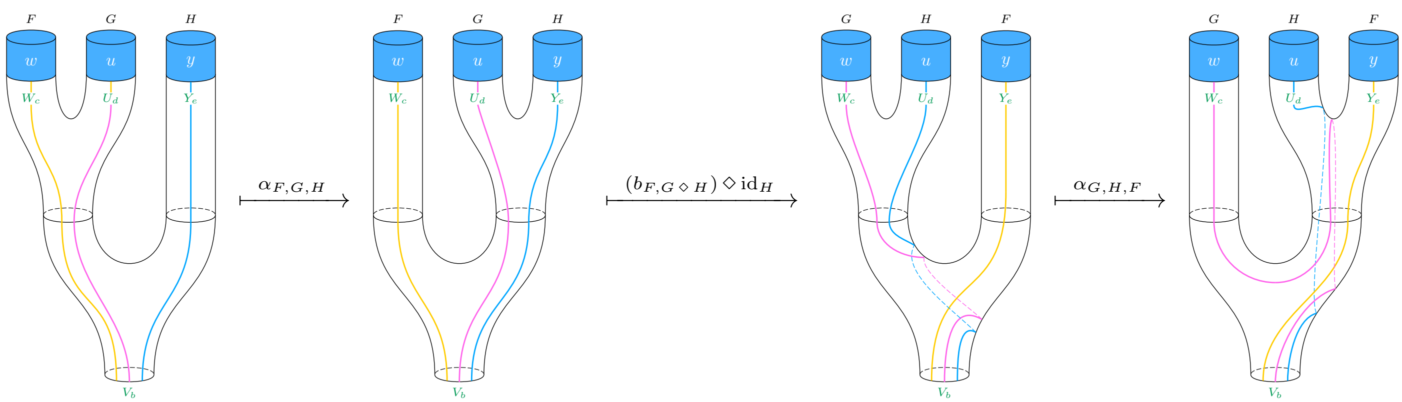
the converse equation follows from a completely mirrored result to establish that:

$$(b^{-1}_{F,G})_{V_b} \circ (b_{F,G})_{V_b} = \mathrm{id}_{(G \diamond F)(V_b)} \tag{3.92}$$

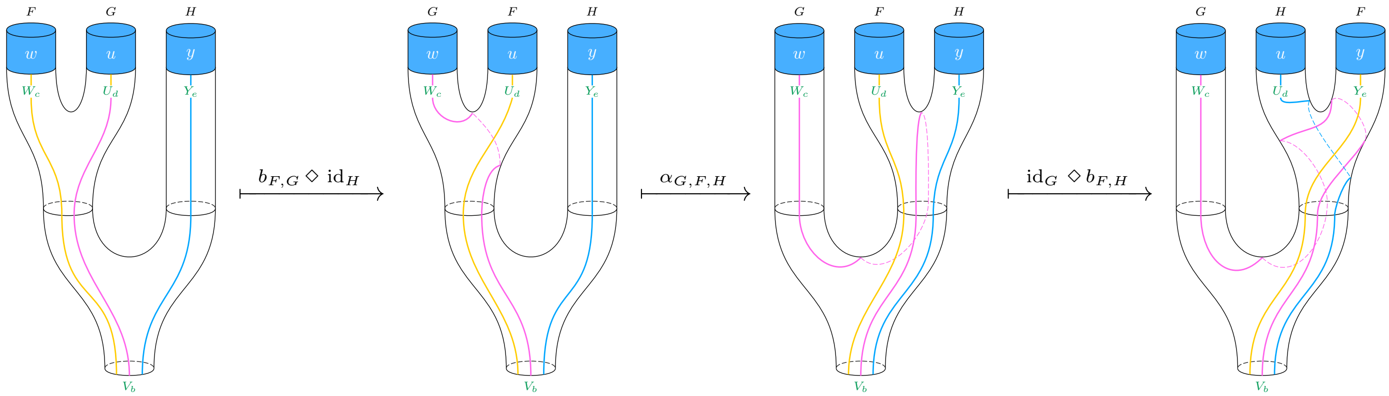
The next step is to evaluate the claim that $b_{F,G}$, satisfy the hexagon equations, for all F and G, modules of the tube category. Picking three arbitrary modules $F, G, H \in \mathrm{Mod}_{\mathrm{Tube}_{\mathcal{C}}(S^1)}$, recall that the hexagon equations are given by:

$$\alpha_{G,H,F} \circ \big(b_{F,G} \diamond \mathrm{id}_H\big) \circ \alpha_{F,G,H} = \big(\mathrm{id}_G \diamond b_{F,H}\big) \circ \alpha_{G,F,H} \circ \big(b_{F,G} \diamond \mathrm{id}_H\big) \tag{3.93}$$

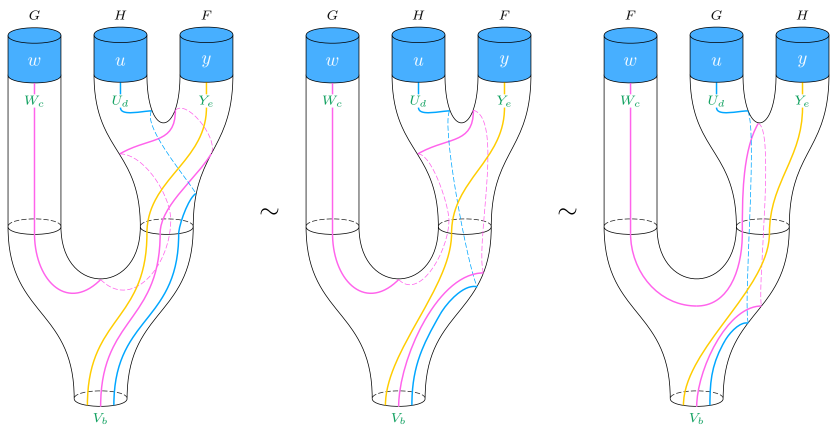
in order to check this holds, we register first the action of the morphism on the left by tracing its effect on a set of fixed colour curves embedded in the "appropriate" manifolds:



followed by using the same analysis with respect to the morphism on the right, we have:



the equality between the action of both morphisms might still not be entirely clear. As such, we proceed with the following clarifying sequence of isotopies:



(3.94)

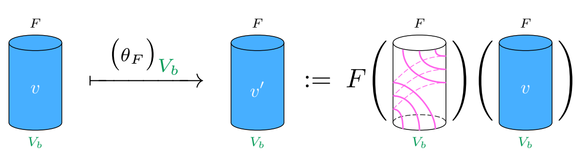
which establishes the proof. $\square$

3.4.1 Ribbon Structure

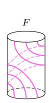
Definition 3.21. (*Ribbon Structure*) Let F be an arbitrary module of the tube category. We define, for every object $V_b \in \text{Obj}\,\mathrm{Mod}_{\mathrm{Tube}_{\mathcal{C}}(S^1)}$ a linear morphism:

$$\left(\theta_F\right)_{V_b} : F(V_b) \longrightarrow F(V_b) \tag{3.95}$$

whose mapping we define as follows:

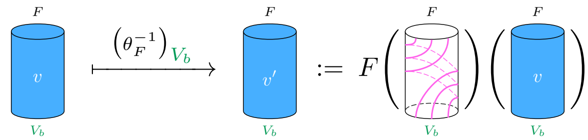


(3.96)

where  is the morphism corresponding to a Dehn-twist. The blue-coloured curves correspond not to a string-net but to the diffeomorphism's action on a select set of embedded 1-manifolds.

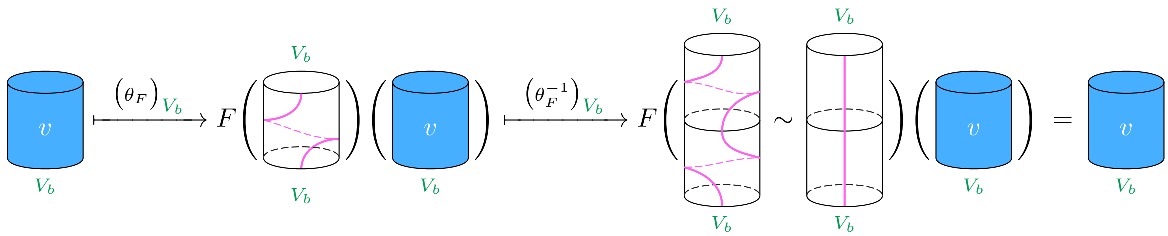
Proposition 3.22. *The collection of linear maps defined in [**?**] form a collection of well-defined, natural isomorphisms that satisfy the balancing axioms. As a consequence it constitutes a ribbon structure for the category of modules of the tube category.*

Proof. To prove this is in fact an isomorphism, we provide an explicit inverse. Let V_b be an object of $\mathrm{Tube}_{\mathcal{C}}(S^1)$, we define:



(3.97)

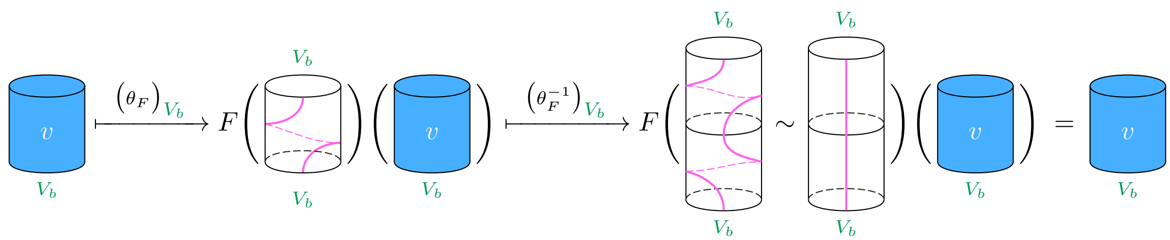
in order to check it is in fact an inverse, we note that:



proving that:

$$\left(\theta_F^{-1}\right)_{V_b} \circ \left(\theta_F\right)_{V_b} = \mathrm{id}_{V_b} \tag{3.98}$$

also:

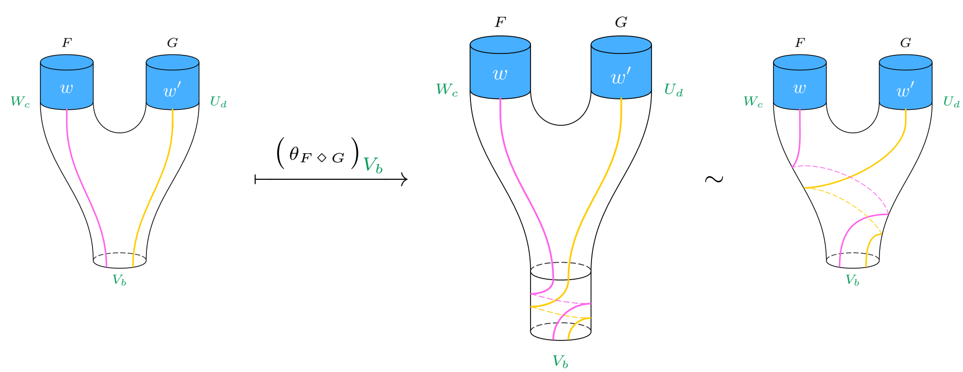


(3.99)

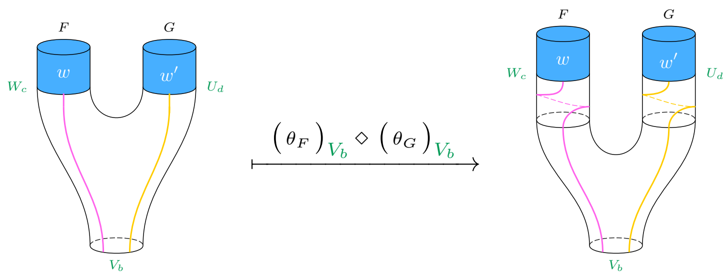
We recall the balancing axioms:

1. $\left(\theta_{F \diamond G}\right)_{V_b} = b_{G,F} \circ b_{F,G} \circ \left[\left(\theta_F\right)_{V_b} \diamond \left(\theta_G\right)_{V_b}\right]$;
2. $\left(\theta_{\mathbf{1}}\right)_{V_b} = \mathrm{id}_{\mathbf{1}(V_b)}$;
3. $\left(\theta_{F^\vee}\right)_{V_b} = \left(\theta_F\right)_{V_b}^\vee$

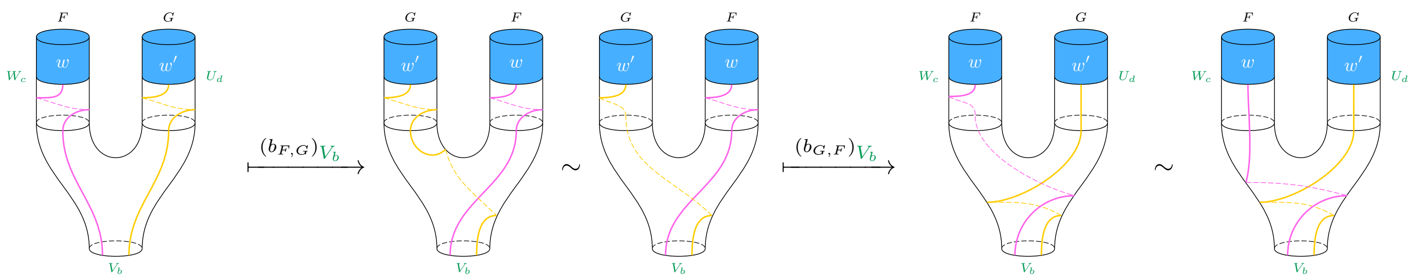
we begin by verifying the first of these points. We note that the left morphism's action can be diagramatically recorded as:



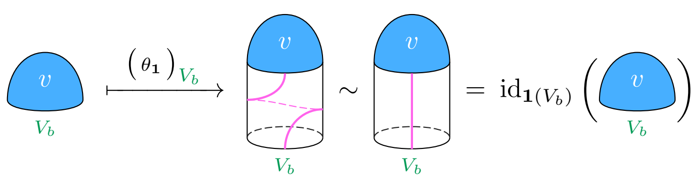
and, on the other hand, the morphism on the right is captured by the following sequence of diagrams. We split the morphism into two and begin by registering the action of the first map on an arbitrary vector:



secondly, the action of the braidings:



which proves the first point of the set of balancing axioms. The second axiom is a consequence of the diagram below:



(3.100)

where the equivalence relation stems from an isotopy that rotates the blue disk cap by a full twist. □

3.5 Modules from the Drinfel'd Center

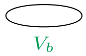
The point of this subsection is to prove the existence of an adjoint equivalence to D, the functor defined in the last subsection, and moreover prove that it is in fact a braided monoidal functor. We note that, througout this sections, both objects and morphisms in the Drinfel'd Center, $\mathcal{Z}(\mathcal{C})$ will be depicted using both green strands and green circular coupons and we will reserve the light blue for objects in $\mathcal{C}$.

Theorem 3.23. *("En-Tubing" Theorem) We define the following functor:*

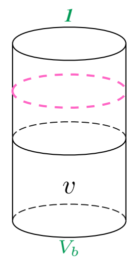
$$E\colon \mathcal{Z}(\mathcal{C}) \longrightarrow \mathrm{Mod}_{\mathcal{T}(S^1)} \tag{3.101}$$

that maps an object $(R, c_{R,-})$ *to the following module:*

$$E(R, c_{R,-})\colon\ \mathrm{Tube}_{\mathcal{C}}(\mathcal{C}) \longrightarrow \mathrm{Vect}$$

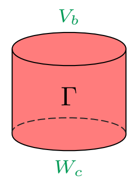
which sends an object 

to the vector space of string-nets defined as follows:

$\frac{1}{\mathcal{D}^2}$ 

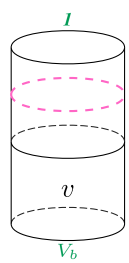
(3.102)

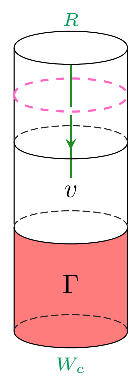
where v is a morphism $\Psi(R) \xrightarrow{v} V_b$ in the tube category. The action of E on morphisms is defined by constructing the linear operators associated to gluing on the bottom boundary of the cylinder. In more descriptive fashion, let:

$\Gamma \;:=$ 

(3.103)

be a morphism in the tube category. The module $E(R, c_{R,-})$, maps Γ to the following linear map:

$\frac{1}{\mathcal{D}^2}$ 

$\xmapsto{E(R,c_{R,-})(\Gamma)}$ $\frac{1}{\mathcal{D}^2}$ 

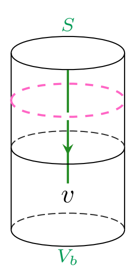
(3.104)

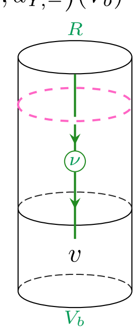
Having finished our description of how E acts on objects of the Drinfel'd Center, we proceed to defining how it acts on morphisms. Let:

$$\nu\colon (R, c_{R,-}) \longrightarrow (Y, d_{Y,-})$$

be morphism in $\nu \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}\big((R, c_{R,-}), (Y, d_{Y,-})\big)$, we then define the following collection of linear maps, indexed by the objects of the tube category:

$$E(\nu)_{V_b} \colon E\big(R, c_{R,-}\big)(V_b) \longrightarrow E\big(Y, d_{Y,-}\big)(V_b) \tag{3.105}$$

$\frac{1}{\mathcal{D}^2}$ 

$\longmapsto$ $\frac{1}{\mathcal{D}^2}$ 

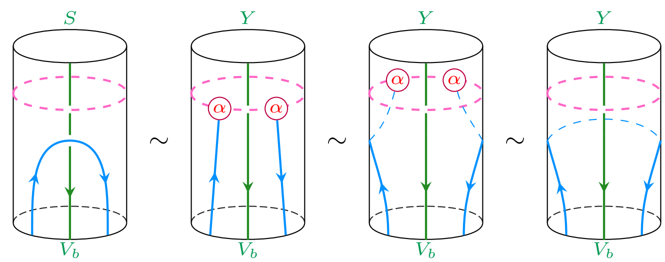
(3.106)

These linear maps are natural and in addition E is a <u>*braided monoidal functor*</u>.

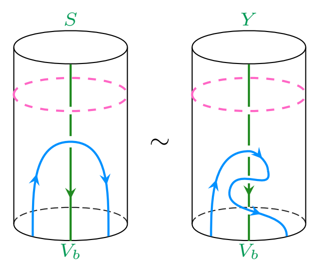
Proof. The theorem contains multiple claims that need checking. The first is that, given an object $(R, c_{R,-})$, the functor $E(R, c_{R,-})$ is in fact well-defined and hence consitutes a module of the tube category. We first note that, given an object of the tube categpry V_b, the vector space:

$$E\big(R, c_{R,-}\big)(V_b) \tag{3.107}$$

can be compared with the vector space of string-nets on the the cylinder with the top circle labeled by $\Psi(R)$ and the bottom cicrle labeled by V_b, the distinction being that it has additional local relations. In particular, we can move strands past the top circle, as the picture below exemplifies:



We can slide blue strands through the green strands as follows:



which means we need additional care when checking whether some maps are well-defined. We now proceed to check that $E(R, c_{R,-})$ is in fact a a functor. We begin by noting that, for every boundary condition V_b, we have that:

$$E\big(R, c_{R,-}\big)(\mathrm{id}_{V_b}) = \mathrm{id}_{E(R,c_{R,-})(V_b)} \tag{3.108}$$

since the map, $E\big(R, c_{R,-}\big)(\mathrm{id}_{V_b})$ corresponds to gluing horizontal identity strands of V_b at the bottom of the string-net and hence leaving it unchanged. In addition, $E\big(R, c_{R,-}\big)$ is tacitly functorial on morphisms:

$$E\big(R, c_{R,-}\big)(\Gamma_1 \circ \Gamma_2) = E\big(R, c_{R,-}\big)(\Gamma_1) \circ E\big(R, c_{R,-}\big)(\Gamma_2) \tag{3.109}$$

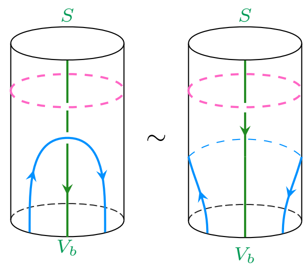
We now shift attention to the action of E on morphisms of the Drinfel'd Center. Given a morphism:

$$\nu\colon (R, c_{R,-}) \longrightarrow (S, d_{S,-}) \tag{3.110}$$

We must check that, given an arbitrary object V_b of the tube category, the linear map:

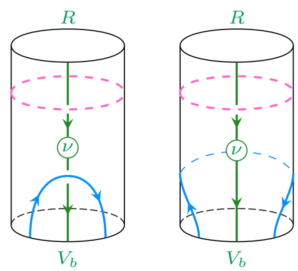
$$E(\nu)_{V_b}\colon E\big(S, d_{S,-}\big)(V_b) \longrightarrow E\big(R, c_{R,-}\big)(V_b) \tag{3.111}$$

is not just well-defined but also natural. We note that that it is well-defined since all of the additional local relations we have previously alluded to, are respected by $E(\nu)_{V_b}$. As an illustration of this, we describe through the example below, how this map is in fact well-defined. We consider two equivalent vectors in $E\big(S, d_{S,-}\big)(V_b)$:



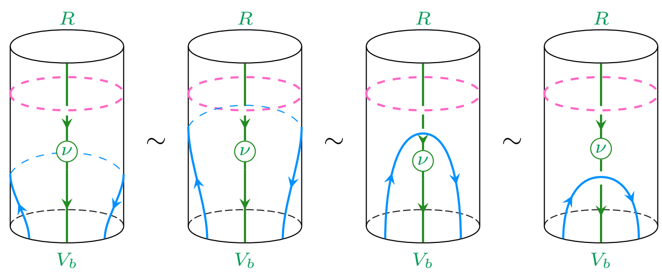
$$(3.112)$$

and their respective image under the action of $E(\nu)_{V_b}$:



$$(3.113)$$

and now we proceed to showcase why they are equivalent, we begin with the diagram on the right:



$$(3.114)$$

where the third relation is allowed only because ν is a morphism of the Drinfel'd Center of $\mathcal{C}$ instead of just an "ordinary" morphism in $\mathcal{C}$. Moreover, this linear map is natural as a consequence of the fact that gluing a string-net at the bottom of the cylinder and then gluing ν at the top is equivalent to first gluing ν at the top and then gluing a string-net at the bottom.

At this point, we have established that E is a well-defined functor. The next step is to check that it is in fact a monoidal functor. In order to do that, we must describe a tensorator, in other words a natural transformation:

$$\phi_{R,S}\colon E\big(R, c_{R,-}\big) \diamond E\big(S, d_{S,-}\big) \Longrightarrow E\big((R, c_{R,-}) \otimes (S, d_{S,-})\big) \tag{3.115}$$

satisfying the following coherence condition for all $(R, c_{R,-}), (S, d_{S,-}), (Q, e_{Q,-}) \in \mathcal{Z}(\mathcal{C})$:

$$\begin{tikzcd}
\big(E(R, c_{R,-}) \diamond E(S, d_{S,-})\big) \diamond E(Q, e_{Q,-}) \arrow[r, "\alpha"] \arrow[d, "\phi_{R,S} \diamond \mathrm{id}_{E(Q)}"'] & E(R, c_{R,-}) \diamond \big(E(S, d_{S,-}) \diamond E(Q, e_{Q,-})\big) \arrow[d, "\mathrm{id}_{E(R)} \diamond \phi_{S,Q}"] \\
E\big((R, c_{R,-}) \otimes_{\mathcal{C}} (S, d_{S,-})\big) \diamond E(Q, e_{Q,-}) \arrow[d, "\phi_{R\otimes S,Q}"'] & E(R, c_{R,-}) \diamond E\big((S, d_{S,-}) \otimes (Q, e_{Q,-})\big) \arrow[d, "\mathrm{id}_{E(R)} \diamond \phi_{S,Q}"] \\
E\big(\big((R, c_{R,-}) \otimes (S, d_{S,-})\big) \otimes (Q, e_{Q,-})\big) \arrow[r, equal] & E\big((R, c_{R,-}) \otimes \big((S, d_{S,-}) \otimes (Q, e_{Q,-})\big)\big)
\end{tikzcd} \tag{3.116}$$

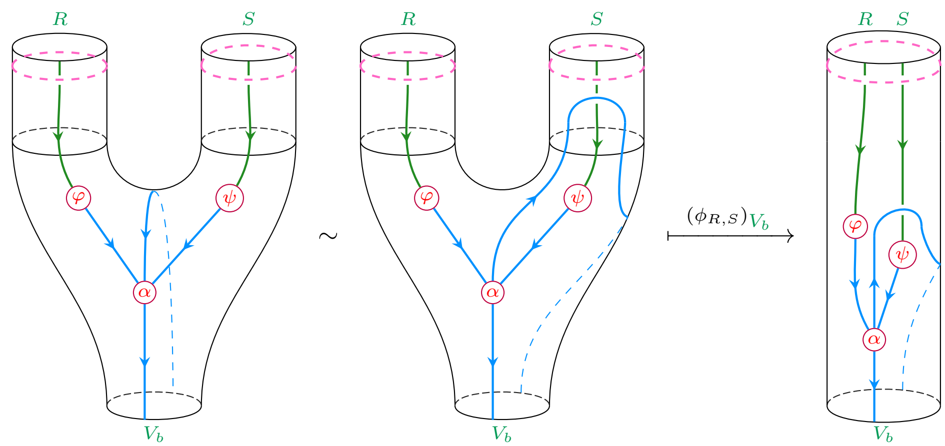
where we have an equality sign on the bottom of the diagram only because we have included, earlier in the thesis, the assumption that $\mathcal{C}$ is a strict rather than a weak monoidal category. Additionally, we need one more piece of data and structure morphism, which we will call unit check and unit constraint:

$$\phi\colon \mathbf{1}_{\mathcal{Z}(\mathcal{C})} \longrightarrow E(\mathbf{1}, \mathrm{id}_{\mathbf{1}}) \tag{3.117}$$

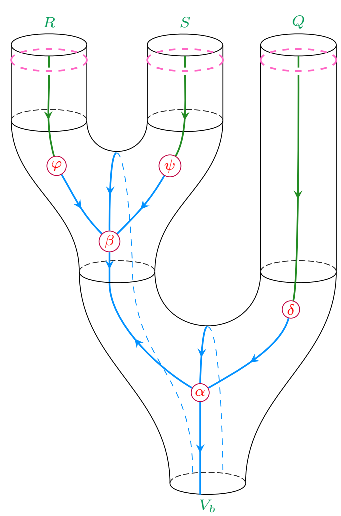
satisfying, for all $(X, c_{X,-}) \in \mathcal{Z}(\mathcal{C})$), the following commutative diagram:

$$\begin{tikzcd} E(R, c_{R,-}) \diamond \mathbf{1} \arrow[r, "1\diamond\phi"] \arrow[d, "\rho_{E(R)}"'] & E(R, c_{R,-}) \diamond E(\mathbf{1}) \arrow[d, "\phi_{R,1}"] \\ E(R, c_{R,-}) \arrow[r, equal] & E\big((R, c_{R,-}) \otimes \mathbf{1}\big) \end{tikzcd} \tag{3.118}$$

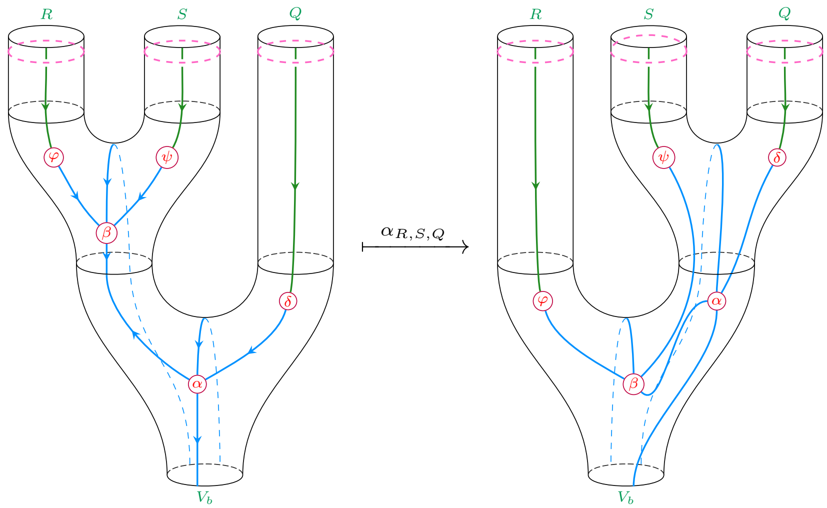
We propose the following linear map for the tensorator:



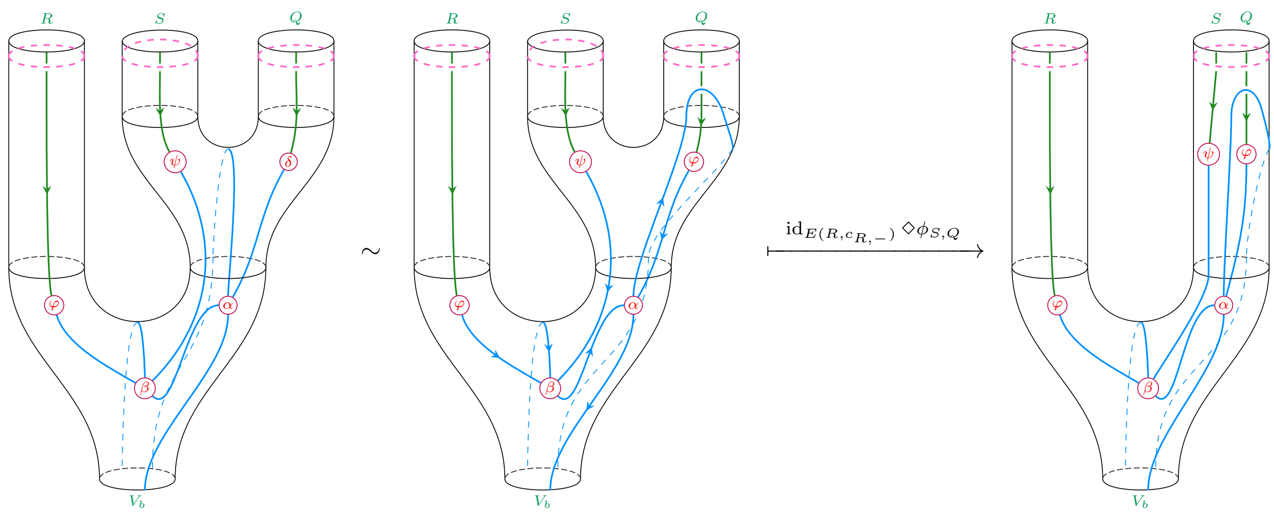
The map is defined by first "clearing" all strands from the saddle region of the pair of pants and then, unambiguously, transform the string-net into one in the cylinder. This is tacitly well-defined and natural since gluing a morphism on the bottom and then applying the tensorator or first applying the tensorator and then gluing the same morphism at the bottom are equal operations. The less obvious part of the proof has to do to with verifying the 3.116 relations. We choose a vector in $\big[\big(E(R, c_{R,-}) \diamond E(S, d_{S,-})\big) \diamond E(Q, e_{Q,-})\big](V_b)$, which we depict as follows:



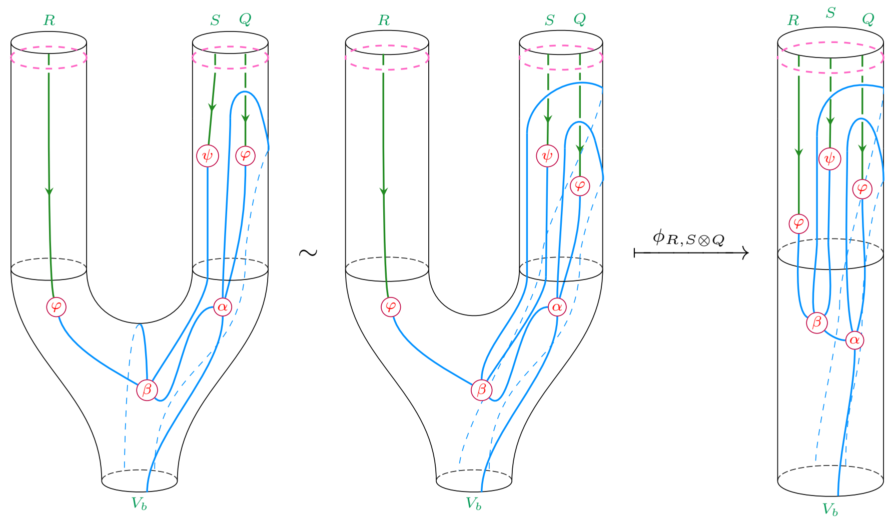
we now proceed to register the action of the sequence of morphisms figuring the top of 3.116. We begin with the action of $\alpha_{R,S,Q}$:



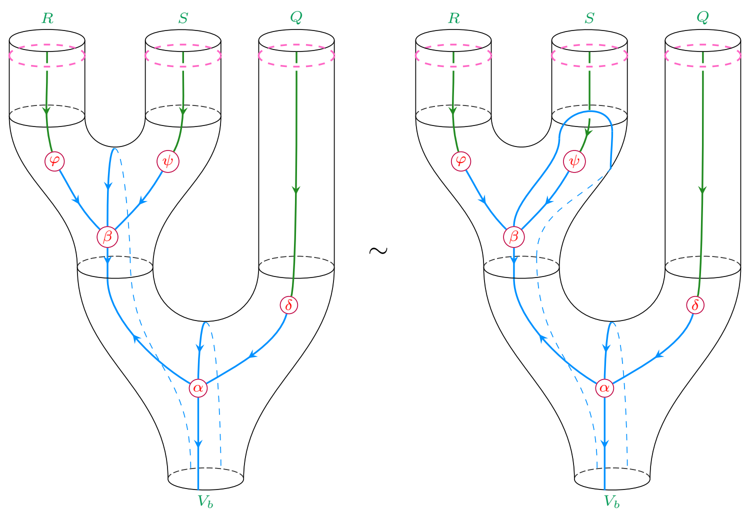
before we apply the tensorator we must transform the diagram on the right by pushing all strands crossing the saddle to the right-hand side of the pair-of-pants:



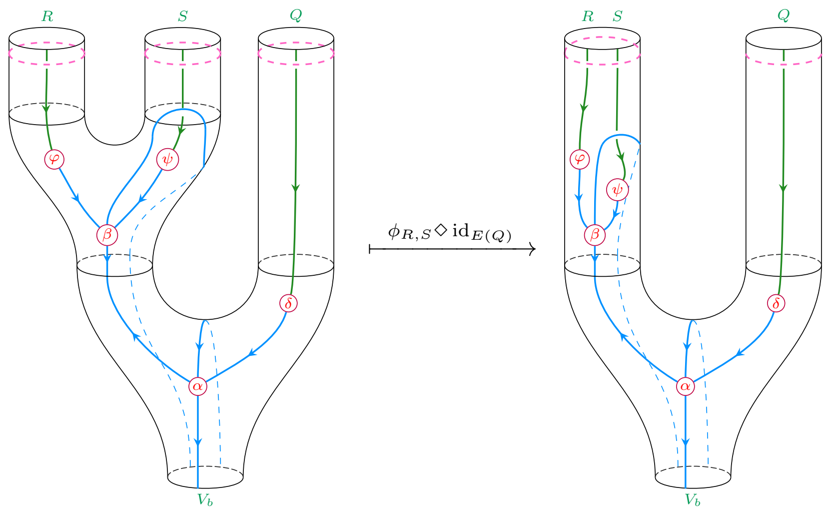
and now repeat the process almost *sine mutatione*, carrying the strand in the saddle into the right leg and then applying the tensorator:



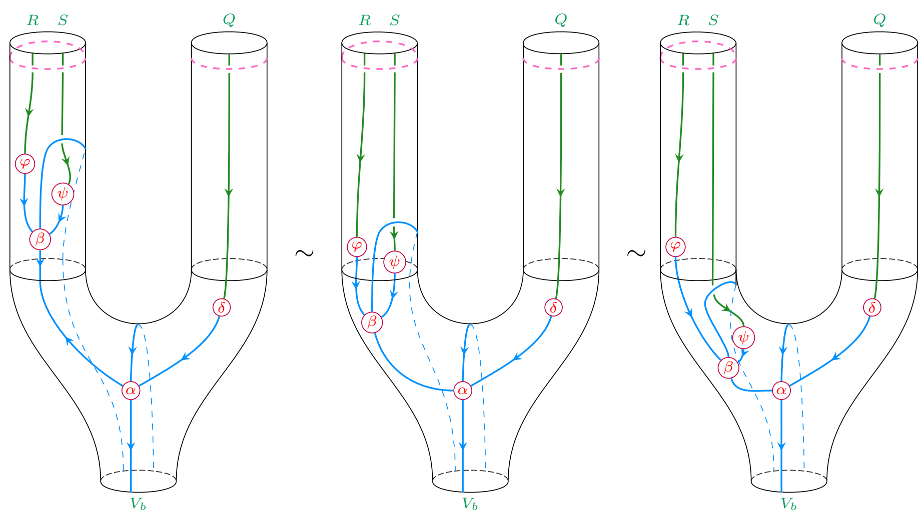
We now restart the circuit, this time following the bottom path. We begin, as we have done, by clearing the strands in the saddle section of the pair-of-pants on the top left:



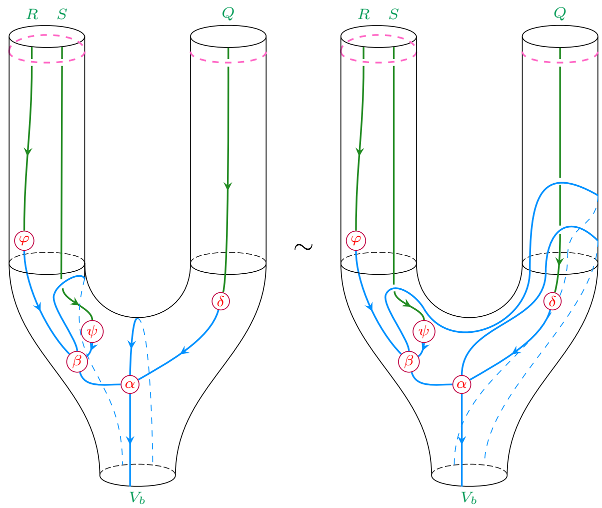
now we apply the tensorator:



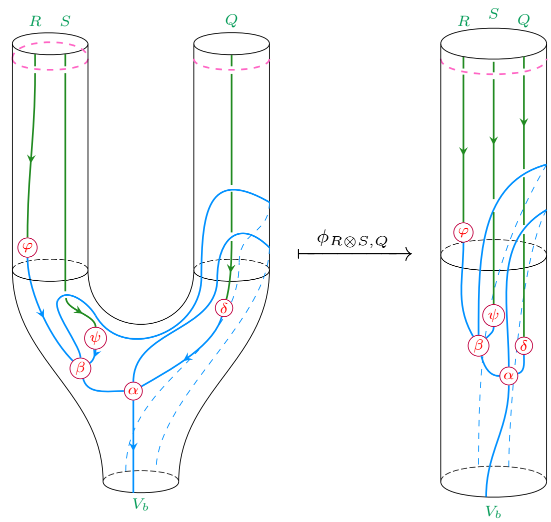
in order to apply the tensorator, as we have done before, we must first move the strands crossing the saddle section into the right leg, but first, in order to best comprehend this step we isotope to a more suitable representative vector:



more descriptively, we have done nothing more than slide the ψ coupon further down the leg into the front of the surface. Now we can, more obviously, apply the change of legs:

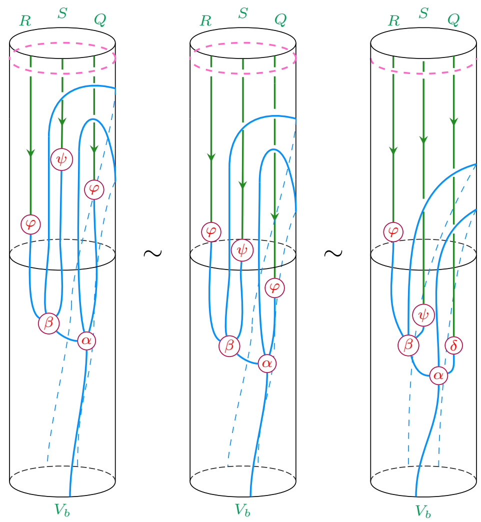


a two-step process was applied. Firstly, we moved the strand that connects α to the waist's boundary and that runs through the saddle section of the surface through the right leg's boundary into the front of the surface and then applied the same to the strand connecting ψ to the waist's boundary and that runs through the saddle section. We can now finally apply the tensorator:



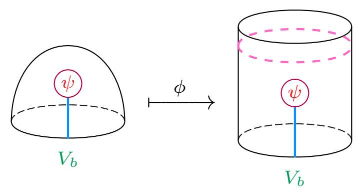
comparing this last diagram with the one at the end of the process followed by going through the top path of diagram

3.116 we have:

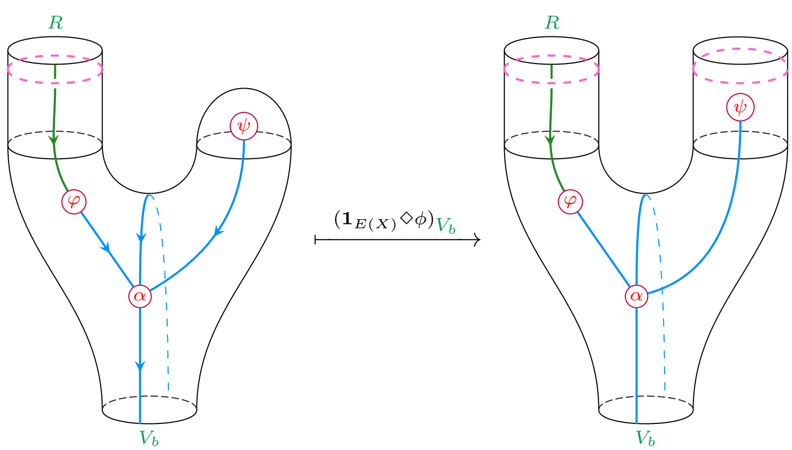


(3.119)

leaving only the structure of the unit constraint to be checked. We propose the following linear map for the unit check:

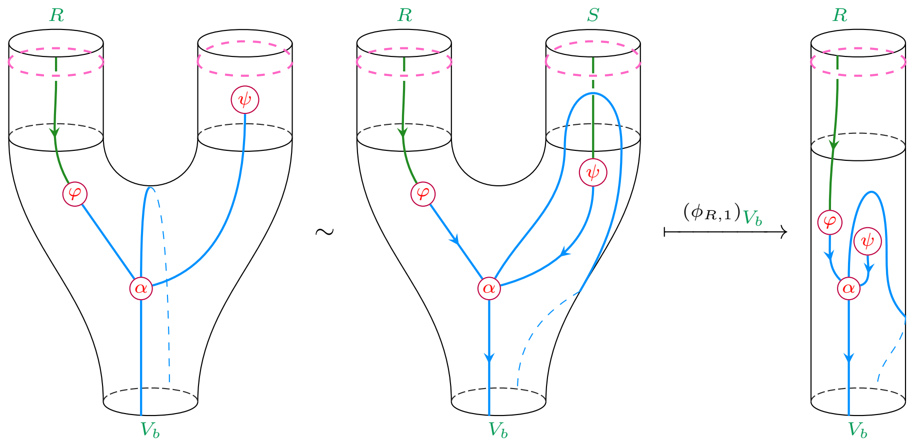


To check the unit constraint, we begin, as we have done, by picking a particular vector in $E(R, c_{R,-}) \diamond \mathbf{1}(V_b)$ and applying the sequence of morphisms in the top of diagram 3.118. The first step is the tensor product of the unit morphism for $E(R, c_{R,-})$ on the left leg with the unit check on the right leg:



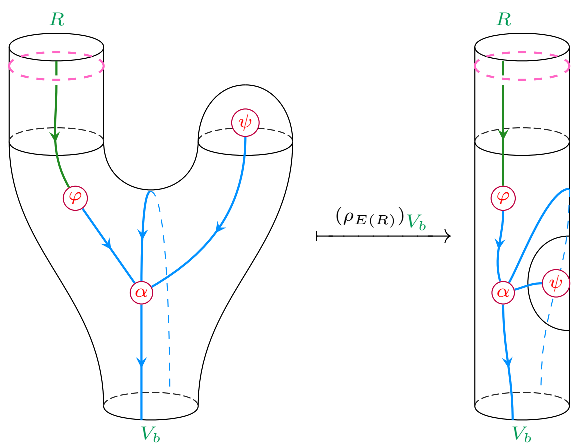
(3.120)

followed by an application of the tensorator and as such, we must first transport all strands intersecting the saddle section to the the right leg:



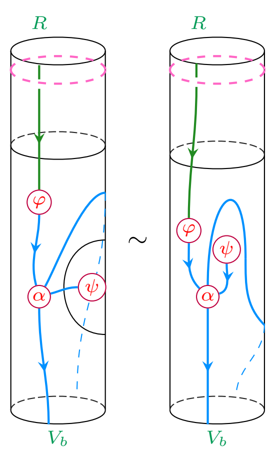
(3.121)

Following the lower path we have:



(3.122)

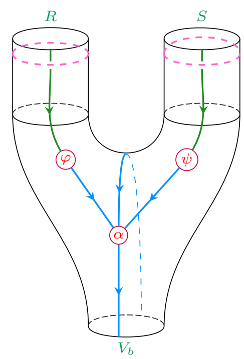
which is easily seen to match the result of the previous process:



finally finishing the proof that E is, in fact, a monoidal functor. The next task is to establish that this monoidal structure, appended to E, is also compatible with the braiding structures of both the Drinfel'd center of $\mathcal{C}$ and the category of modules of the tube category. We recall that this compatibility amounts to the commutativity of the following diagram:

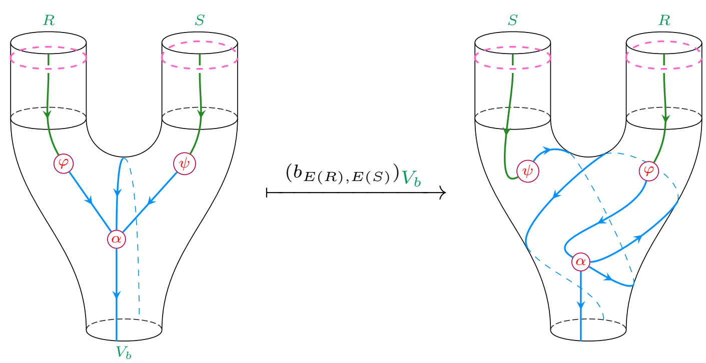
$$\begin{tikzcd}
E(R, c_{R,-}) \diamond E(S, d_{S,-}) \arrow[r, "b_{E(R),E(S)}"] \arrow[d, "\phi_{R,S}"'] & E(S, c_{S,-}) \diamond E(R, c_{R,-}) \arrow[d, "\phi_{S,R}"] \\
E\big((R, c_{R,-}) \otimes (S, d_{S,-})\big) \arrow[r, "E(c_{S,R})"] & E\big((S, d_{S,-}) \otimes (R, c_{R,-})\big)
\end{tikzcd} \tag{3.123}$$

To verify this, we pick the following vector in $E(R, c_{R,-}) \diamond E(S, d_{S,-})(V_b)$:



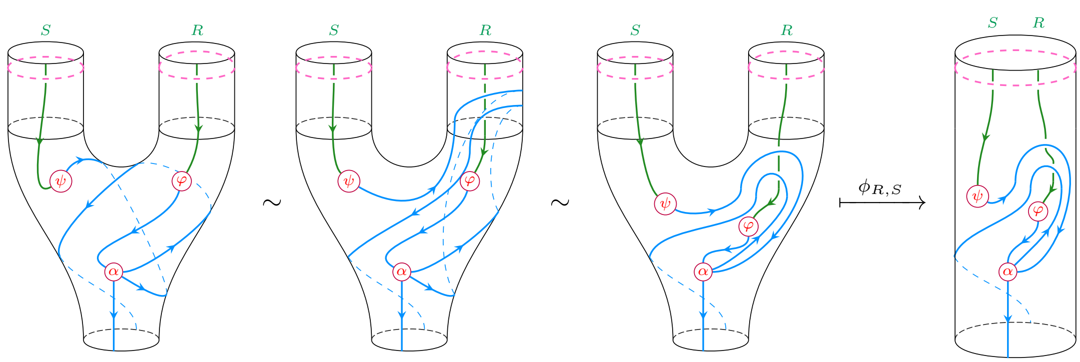
(3.124)

and apply the braiding morphism:



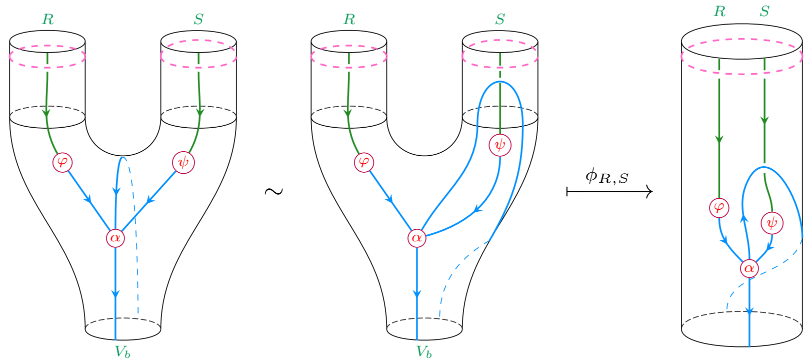
(3.125)

before applying the tensorator, we clear all strands intersecting the saddle section and to make the tensorator's action more explicit we first push the strands to the front of the picture:

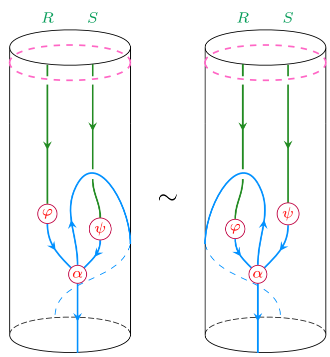


(3.126)

Now we trace the action of the sequence of morphisms in the bottom of the diagram 3.132. We begin by applying the tensorator:

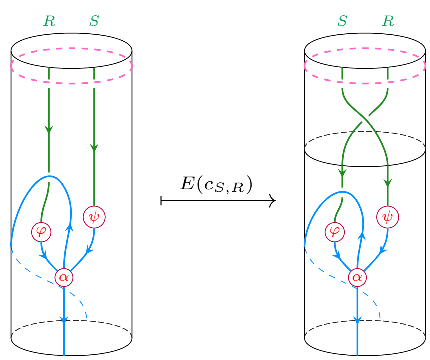


before proceeding, we shift the strand in the right of the tube to the left:

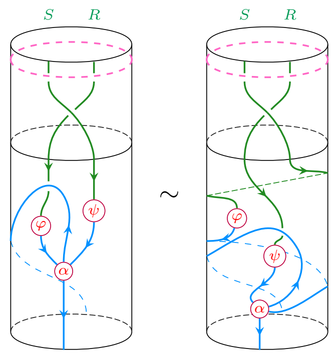


(3.127)

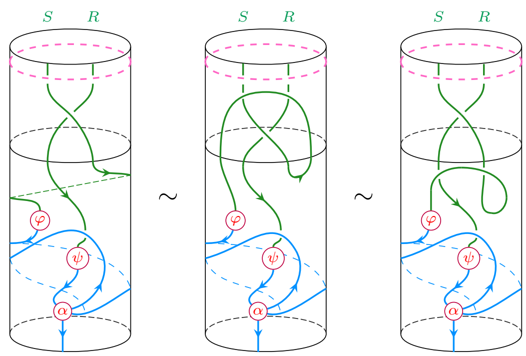
now we apply $E(c_{S,R})$:



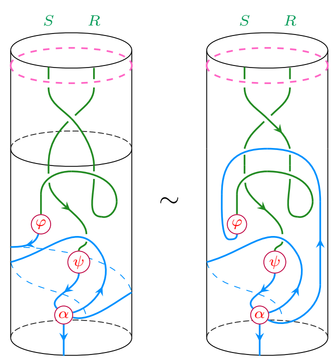
by gluing $c_{S,R}$ on top of the tube. At this point it is not at all clear that this output matches in anyway with the one produced from the top path of diagram 3.132. To clarify this point we will strategically move around some of the pieces. We begin sliding the ψ coupon to the front right of the picture and slide φ coupon around the back of the cylinder until it appears on the left-front side of the picture:



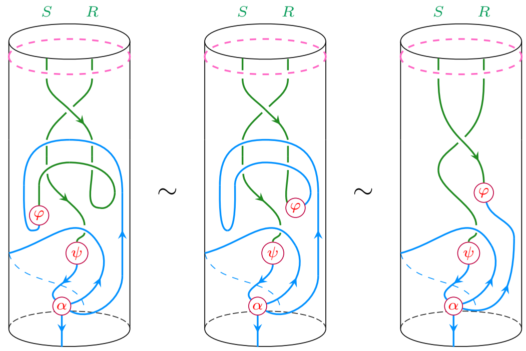
after doing this, we pull, across the top boundary circle, the green strand of the object R and proceed to use the hexagon equations in order to pull this same strand further below:



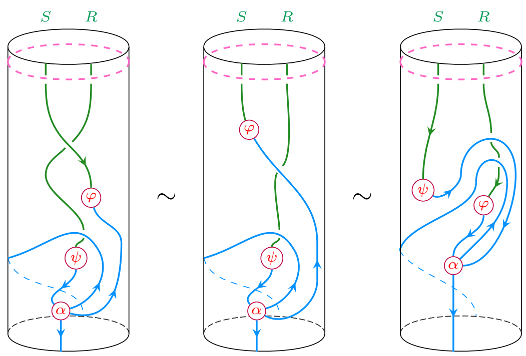
and repeat, *sine mutatione*, the same sequence of operations for the blue strand that runs around the back of the cylinder and that connects the α coupon to the ψ coupon:



Next, we make use of the fact that the half-braidings in the Drinfel'd center are natural transformations in order to slide φ from the front-left to front-right side of the picture and follow that move by pulling the blue strand to a "least curved" position by pulling it to its straightest path:



having almost finished the proof we still need to make use of the naturality of the half-braiding to push it to the left side of the picture and match it with with our target:



$\square$

Before we check that E and D are inverse to each other, we must still solve one problem. We have defined D on $\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$ and E's image lies $\mathrm{Mod}_{\mathcal{T}(S^1)}$. To reckon with this discrepancy we must first prove the following result:

Theorem 3.24. *Let* $(R, c) \in \mathcal{Z}(\mathcal{C})$. *We have that:*

$$E(X, c) \circ \Psi \simeq \mathrm{Hom}_{\mathcal{C}}(R, -) \tag{3.128}$$

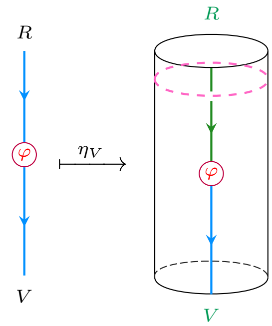
where Ψ *is the canocnical inclusion functor* $\Psi \colon \mathcal{C} \longrightarrow \mathrm{Tube}_{\mathcal{C}}(S^1)$.

Proof. The proof consists in the explicit construction of a natural isomorphism. We define

$$\mathrm{Hom}_{\mathcal{C}}(R, -) \overset{\eta}{\Longrightarrow} E(X, c) \tag{3.129}$$

in the following manner. Given an object V of $\mathcal{C}$, we set:

$$\eta_V \colon \operatorname{Hom}_{\mathcal{C}}(R, V) \longrightarrow E(R, c)(\Psi(V)) \tag{3.130}$$



(3.131)

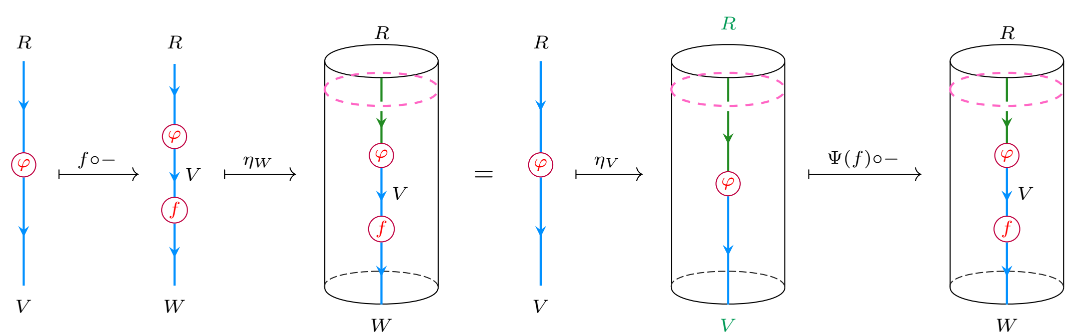
which is a well-defined linear map. In order for η_V to be what we are looking for, we must provide a proof for each item in the following list:

1. η_V is surjective;
2. η_V is injective;
3. η_V is natural

We begin with simplest of these assertions, namely 1 and 3. To see that η_V is natural we note that, for every $f \in \operatorname{Hom}_{\mathcal{C}}(V, W)$, commutativity of the following diagram:

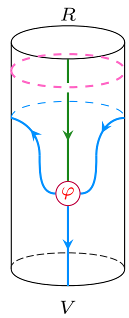
$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}}(R, V) \arrow[r, "\eta_V"] \arrow[d, "f\circ -"'] & E(R, c)(\Psi(V)) \arrow[d, "\Psi(f)\circ -"] \\
\operatorname{Hom}_{\mathcal{C}}(R, W) \arrow[r, "\eta_W"'] & E(R, c)(\Psi(W))
\end{tikzcd} \tag{3.132}$$

reduces to the obvious equality:



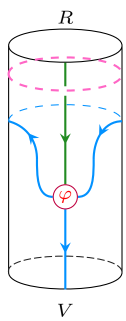
(3.133)

hence proving point 1. To check that η_V is surjective we make use of the result in beginning of the section to claim that, given an arbitrary element in $E(R, c)(\Psi(V))$, we can pick a non-unique representative of its equivalence class with the following form:

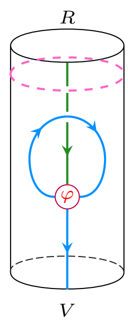


(3.134)

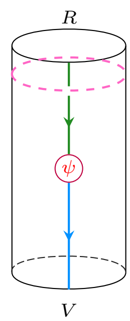
which we can transform, in the quotient, into the following representative:



$\sim$



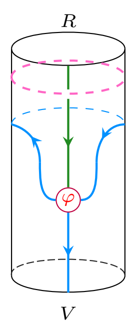
$:=$



(3.135)

immediately ensuring that there exists a morphism $\psi \in \mathrm{Hom}_{\mathcal{C}}(R, V)$ in $\mathcal{C}$ such that:

$\eta_V(\psi) \ =$



(3.136)

establishing surjectivity of η_V and clearing up item 2. Injectivity is less obvious and more laborious to prove. Given a pair of equivalent vectors in $E(R, c)(\Psi(V))$ we note there exists a finite number of disks, properly embedded in one of the cylinders and a collection of local relations attached to each circle that transform one of the diagrams into the other. Applying the same procedure as the one above, used to prove surjectivity, we convert both diagrams into ones of the form exemplified in equation 3.136. After doing this, it turns out the finite set of disks and local relations that, when evaluated, transform one representative vector into the other, are just shifted in the process of standardizing and hence will obviously evaluate to the same morphism in $\mathcal{C}$ safe, at least at a glance, for the disks that, after standardizing, intersect the green strand running from the top of the cylinder. To ensure that even these local relations remain valid when we apply this process, we need just to recall that: the local relations in a disk can be split into ones enforced within both of its hemispheres and all the isotopies that shift across its gluing interval — this come from gluing rules along intervals. In the case of the local relations that are mapped to disks intersecting the green strand, this is true since all strands and all morphisms can slide past a green strand via the graphical calculus in $\mathcal{Z}(\mathcal{C})$, hence proving that no local relations are destroyed and ensuring injectivity of η_V. The post-composition follows directly from this result. $\square$

To finish the result we proposed in the beginning of this section we have, still to prove, that the functors E and D are inverse to each other.

Theorem 3.25. *Let $\mathcal{C}$ be a spherical fusion category. The category of modules of $\mathrm{Tube}_{\mathcal{C}}(S^1)$ is contravariantly braided monoidal equivalent to the Drinfel'd center $Z(\mathcal{C})$.*

Proof. In order to avoid this proof into turning into an excessively complicated ordeal, we first provide an itemized outline of what must be checked in order to prove the result:

1. Guarantee that $E \circ D$ is the identity functor for $\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$:
   (a) Given a module F, and given some $\{V_b\}$ boundary condition, we have that
   $$[E \circ D](F)(\{V_b\}) = F(\{V_b\}), \tag{3.137}$$
   and given some morphism Γ in the tube category, we have that:
   $$[E \circ D](F)(\Gamma) = F(\Gamma). \tag{3.138}$$
   Hence implying an equality of modules in $\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$;
   (b) Given a morphism of modules i.e. a natural transformation $\eta\colon F \Longrightarrow G$, we must check that:
   $$[E \circ D](\eta\colon F \Longrightarrow G) = \eta. \tag{3.139}$$
2. Guarantee that $D \circ E$ is the identity functor for $\mathcal{Z}(\mathcal{C})$:
   (a) Given an object of the center (R, c), verify that if:
   $$[D \circ E](R, c) = (S, d) \tag{3.140}$$
   then $R = S$ and that $d = c$;
   (b) Given a morphism in the Drinfel'd center of $\mathcal{C}$
   $$\varphi\colon (R, c) \longrightarrow (S, d), \tag{3.141}$$
   we must check that
   $$[D \circ E]\big(\varphi\colon (R, c) \longrightarrow (S, d)\big) = \varphi \tag{3.142}$$

1(a) — To check this, we note that since F is in $\overline{\mathrm{Mod}}_{\mathcal{T}(S^1)}$ there exists an object $R \in \mathcal{C}$, such that

$$F(\Psi(V)) = \mathrm{Hom}_{\mathcal{C}}(R, V).$$

Moreover, let c be the half-braiding, such that $(R, c) = D(F)$. Making use of Theorem 8.16, we have that,

$$E(R, c)(\Psi(V)) = \mathrm{Hom}_{\mathcal{C}}(R, V) = F(\Psi(V)).$$

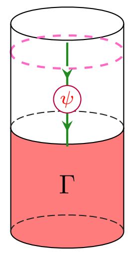
We are then left to check that $[E \circ D](F)(\Gamma) = F(\Gamma)$. Let $\Gamma\colon \{V_b\} \longrightarrow \{Z_c\}$ be a a morphism and let $\{V_b\} = \Psi(V)$ and $\{Z_c\} = \Psi(Z)$. Without losing generality, we assume Γ can be factored as follows:

$$\Gamma = \Psi(\varphi) \circ \beta_{V,W} \tag{3.143}$$

where $\varphi \in \mathrm{Hom}_{\mathcal{C}}(W^* \otimes V \otimes W \otimes, Z)$. From our definition of E and D, together with Proposition 3.24, we know that:

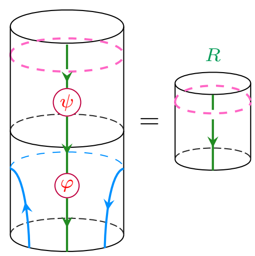
$$[E \circ D](F)(\Gamma) = E(R, c)(\Gamma)\colon\ \mathrm{Hom}_{\mathcal{C}}(R, V) \longrightarrow \mathrm{Hom}_{\mathcal{C}}(R, Z) \tag{3.144}$$
$$\psi \longmapsto \Gamma \circ \Psi(\psi), \tag{3.145}$$

using the identification of String-Net vectors in $E(R,c)(\{Z_c\})$ with morphisms in $\mathrm{Hom}_{\mathcal{C}}(R,Z)$, constructed in Proposition 3.24. The mapping looks as follows:



(3.146)

If we take the diagram above and isotope it we get:

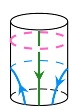
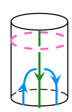


(3.147)

So we can rewrite the mapping $E(R,c)(\Gamma)$ as:

$$E(R,c)(\Gamma)(\psi) = \Psi(\psi \circ (\mathrm{id}_{W^*} \otimes \psi \otimes \mathrm{id}_W)) \circ \beta_{R,W}, \tag{3.148}$$

and since

$\beta_{R,W}$ =  =  = $\Psi(\tilde{c}_{R,W})$, (3.149)

equation 3.148 can be rewritten

$$E(R,c)(\Gamma)(\psi) = \Psi(\psi \circ (\mathrm{id}_{W^*} \otimes \psi \otimes \mathrm{id}_W)) \circ \beta_{R,W} = \Psi(\psi \circ (\mathrm{id}_{W^*} \otimes \psi \otimes \mathrm{id}_W)) \circ \Psi(\tilde{c}_{R,W}) \tag{3.150}$$

$$= \Psi(\psi \circ (\mathrm{id}_{W^*} \otimes \psi \otimes \mathrm{id}_W) \circ \tilde{c}_{R,W}). \tag{3.151}$$

$$\psi \circ (\mathrm{id}_{W^*} \otimes \psi \otimes \mathrm{id}_W) \circ \tilde{c}_{R,W} = \psi \circ ((\mathrm{id}_{W^*} \otimes \psi \otimes \mathrm{id}_W) \circ F(\beta_{V,W})(\mathrm{id}_V)). \tag{3.152}$$

Where we just rewrote $\tilde{c}_{R,W}$ using its definition. Now, we can apply the result in 3.32, to obtain:

$$\psi \circ ((\mathrm{id}_{W^*} \otimes \psi \otimes \mathrm{id}_W) \circ F(\beta_{V,W})(\mathrm{id}_V)) = \psi \circ F(\beta_{V,W})(\psi). \tag{3.153}$$

Recalling that post-composition with ψ is the action of F on morphisms in the inclusion and applying the functoriality of F, we have:

$$\psi \circ F(\beta_{V,W})(\psi) = F(\Psi(\varphi)) \circ F(\beta_{V,W})(\psi) = F(\Psi(\varphi) \circ \beta_{V,W})(\psi). \tag{3.154}$$

Recalling the factorization of Γ in 3.143, we have that:

$$F(\Psi(\varphi) \circ \beta_{V,W})(\psi) = F(\Gamma)(\psi). \tag{3.155}$$

Hence, establishing that:

$$F(\Gamma) = [E \circ D](F)(\Gamma). \tag{3.156}$$

1(b) — Assume that R and S are representing objects for F and G, respectively. Recall from **??** that given a morphism η between such modules F and G, we define

$$D(\eta) = \eta_R(\mathrm{id}_R) \in \mathrm{Hom}_{\mathcal{Z}(\mathcal{C})}(S, R).$$

Moreover, note that $[E \circ D](\eta\colon F \implies G)$ is, by assumption, a morphism of representable functors when canonically restricted to $\mathcal{C}$. Therefore, given some $\{V_b\} \in \mathrm{Tube}_{\mathcal{C}}(S^1)$, such that $\Psi(V) = \{V_b\}$, and given an arbitrary morphism $\varphi \in \mathrm{Hom}_{\mathcal{C}}(R, V) = F(\{V_b\})$, we check that:

$$[E \circ D](\eta)_{\{V_b\}}(\varphi) = E\big(\eta_{\Psi(R)}(\mathrm{id}_R)\big)(\varphi) = \varphi \circ \eta_{\Psi(R)}(\mathrm{id}_R), \tag{3.157}$$

where in the first equality we just apply the definition of D to natural transformations and in the second equality we apply the result proved in Theorem **??** ensuring that E's action on morphisms is just pre-composition with $\eta_{\Psi(R)}(\mathrm{id}_R)$. From the discussion in **??**, on the Yoneda lemma, it directly follows that

$$\varphi \circ \eta_{\Psi(R)}(\mathrm{id}_R) = \eta_{\Psi(V)}(\varphi),$$

hence proving that

$$[E \circ D](\eta)_{\{V_b\}}(\varphi) = \eta_{\Psi(V)}(\varphi) = \eta_{\{V_b\}}(\varphi),$$

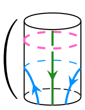
which checks point 1(a).

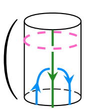
2(a) — It is a direct corollary of Theorem 8.16 that, if $(S, d) := [D \circ E](R, c)$, then

$$S = R.$$

The rest of the verification follows from the equality

$$\tilde{d}_{R,V} = E(R,c)(\beta_{R,V})(\mathrm{id}_R) \; = \; \Psi\Bigg(\quad\Bigg) \; = \; \Psi\Bigg(\quad\Bigg) \; = \; \tilde{c}_{R,V} \tag{3.158}$$





2(b) We again reiterate that $D(\eta) = \eta_S(\mathrm{id}_S)$. As such,

$$[D \circ E](\varphi) = E(\varphi)_{\Psi(S)}(\mathrm{id}_S) \tag{3.159}$$

from Theorem **??**, we know that $E(\varphi)$ is given as pre-composition with φ. Therefore, we have that

$$[D \circ E](\varphi) = E(\varphi)_{\Psi(S)}(\mathrm{id}_S) = \mathrm{id}_S \circ \varphi = \varphi \tag{3.160}$$

as was required. □

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